Solutions to the Exercises in Structural Equation Modeling

Dr. Kerstin Hesse

Email: kerstin.hesse@hhl.de; *Phone:* +49 (0)341 9851-820; *Office:* HHL Main Building, Room 115A

HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany

Doctoral Program at HHL, June 1-2, 2012

Outline & Table of Contents

This document contains the solutions to the exercises in the course material (on the slides).

I will upload this document after the end of the course, so that you have all the solutions for the assignment.

Structural Equation Modeling

Solutions to Topic 1: Revision of Linear Algebra and Variance/Covariance

Dr. Kerstin Hesse

Email: kerstin.hesse@hhl.de; *Phone:* +49 (0)341 9851-820; *Office:* HHL Main Building, Room 115A

HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany

Doctoral Program at HHL, June 1-2, 2012

Ex. 1.1: Matrix Multiplication

Execute the *matrix multiplication*

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

<u>Solution:</u>

$$\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot 1 + 1\cdot 4 + (-1)\cdot 7 & 1\cdot 2 + 1\cdot 5 + (-1)\cdot 8 & 1\cdot 3 + 1\cdot 6 + (-1)\cdot 9 \\ (-1)\cdot 1 + 1\cdot 4 + 0\cdot 7 & (-1)\cdot 2 + 1\cdot 5 + 0\cdot 8 & (-1)\cdot 3 + 1\cdot 6 + 0\cdot 9 \\ 0\cdot 1 + (-2)\cdot 4 + 1\cdot 7 & 0\cdot 2 + (-2)\cdot 5 + 1\cdot 8 & 0\cdot 3 + (-2)\cdot 6 + 1\cdot 9 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & 0 \\ 3 & 3 & 3 \\ -1 & -2 & -3 \end{pmatrix}$$

Ex. 1.2: Determinants of 2×2 and 3×3 Matrices

Compute the *determinants* of the following two matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

<u>Solution:</u>

$$\det(\mathbf{A}) = 1 \cdot 4 - 3 \cdot 2 = 4 - 6 = -2$$

and

$$\begin{aligned} \det(\mathbf{B}) &= 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 7 \cdot 5 \cdot 3 - 8 \cdot 6 \cdot 1 - 9 \cdot 4 \cdot 2 \\ &= 45 + 84 + 96 - 105 - 48 - 72 \\ &= 225 - 225 = 0. \end{aligned}$$

Ex. 1.3: General Formula for the Determinant

Use the *expansion with respect to the first column formula* from page 17 of the lecture slides to compute the *determinant* of

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

<u>Solution:</u> We expand with respect to the first column:

$$\begin{aligned} \det(\mathbf{B}) &= (-1)^{1+1} \cdot 1 \cdot \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} + (-1)^{2+1} \cdot 4 \cdot \det \begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} \\ &\quad + (-1)^{3+1} \cdot 7 \cdot \det \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix} \\ &= [5{\cdot}9 - 8{\cdot}6] - 4 \cdot [2{\cdot}9 - 8{\cdot}3] + 7 \cdot [2{\cdot}6 - 5{\cdot}3] \\ &= [45 - 48] - 4 \cdot [18 - 24] + 7 \cdot [12 - 15] = -3 + 24 - 21 = 0. \end{aligned}$$

Ex. 1.4: Eigenvectors and Eigenvalues

Consider the *symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 3 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix},$$

1. Compute the *eigenvalues* $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and the *corresponding eigenvectors* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ of $\mathbf{A}$ (where $\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i$, $i = 1, 2, 3$).
2. Find an *orthogonal matrix* $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\,\mathbf{A}\,\mathbf{S} = \mathbf{S}'\,\mathbf{A}\,\mathbf{S} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \qquad (1)$$

with $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Execute the matrix multiplication in (1) to verify that you have chosen $\mathbf{S}$ correctly.

Ex. 1.4: Eigenvectors and Eigenvalues

Solution of Part 1: We find the zeros/roots the characteristic polynomial

$$p(\mathbf{A},\lambda) = \det\big(\lambda\,\mathbf{I} - \mathbf{A}\big)$$

of $\mathbf{A}$: Using the rule for the determinant of 3×3 matrices yields

$$\begin{aligned}
p(\mathbf{A},\lambda) &= \det\big(\lambda\,\mathbf{I} - \mathbf{A}\big) = \det\begin{pmatrix} \lambda - \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & \lambda - 3 & 0 \\ -\frac{1}{2} & 0 & \lambda - \frac{3}{2} \end{pmatrix} \\
&= \left(\lambda - \frac{3}{2}\right)^2 (\lambda - 3) - \left(-\frac{1}{2}\right)^2 (\lambda - 3) \\
&= \left(\lambda^2 - 3\,\lambda + \frac{9}{4} - \frac{1}{4}\right)(\lambda - 3) \\
&= \big(\lambda^2 - 3\,\lambda + 2\big)(\lambda - 3) = (\lambda - 1)\,(\lambda - 2)\,(\lambda - 3),
\end{aligned}$$

and we see that the *eigenvalues* are $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 1$.

Ex. 1.4: Eigenvectors and Eigenvalues

To *find the eigenvectors*, we solve $(\lambda_j \mathbf{I} - \mathbf{A})\,\mathbf{x}_j = \mathbf{0}$ for $j = 1, 2, 3$.

More precisely, for each value of λ_i we have to solve the linear system

$$\begin{pmatrix} \lambda_i - \frac{3}{2} & 0 & -\frac{1}{2} \\ 0 & \lambda_i - 3 & 0 \\ -\frac{1}{2} & 0 & \lambda_i - \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad (2)$$

where the eigenvector $\mathbf{x}_i$ is denoted by $\mathbf{x}_i = (x, y, z)'$.

We note that this is the same as solving the linear system

$$\begin{array}{rcccccl} \left(\lambda_i - \frac{3}{2}\right) \cdot x & + & 0 \cdot y & - & \frac{1}{2} \cdot z & = & 0 \\ 0 \cdot x & + & (\lambda_i - 3) \cdot y & + & 0 \cdot z & = & 0 \\ -\frac{1}{2} \cdot x & + & 0 \cdot y & + & \left(\lambda_i - \frac{3}{2}\right) \cdot z & = & 0 \end{array} \qquad (3)$$

and it is also equivalent to solving $\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i$. However, it is *more convenient (less computational work!) to use a system with a zero vector on the right-hand side*, and so we prefer to work with (2) or (3).

Ex. 1.4: Eigenvectors and Eigenvalues

The mathematically economical way to solve such a linear system is to write is as an *augmented matrix* $(\mathbf{A} - \lambda_i\, \mathbf{I}|\mathbf{0})$, more explicitly:

$$\left(\begin{array}{ccc|c} \lambda_i - \frac{3}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \lambda_i - 3 & 0 & 0 \\ -\frac{1}{2} & 0 & \lambda_i - \frac{3}{2} & 0 \end{array} \right). \tag{4}$$

You can do this for any linear system $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ and would have $(\mathbf{A}|\mathbf{b})$; the last column contains the right-hand side $\mathbf{b}$ of the linear system.

On (4) (and more generally on $(\mathbf{A}|\mathbf{b})$) we can now perform *elementary row operations* (*important:* also apply the operation to the last column!):

- multiply/divide a row by a real number
- add/subtract a row from another row.
- swap two rows
- combinations: add/subtract a multiple of a row to/from another row

Ex. 1.4: Eigenvectors and Eigenvalues

For $\lambda_1 = 3$, we first add the third row to the first row.

$$\left(\begin{array}{ccc|c} \frac{3}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{3}{2} & 0 \end{array}\right)$$

Subsequently we add $1/2$ times the new first row to the third row. Then we divide the new third row by 2.

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

Finally, we subtract the new third row from the new first row.

$$\Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Leftrightarrow \mathbf{x}_1 = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha \in \mathbb{R}. \qquad (5)$$

Setting $\mathbf{x}_3 = (x, y, z)'$ we get, $x = 0$, $z = 0$, $y = \alpha$ for any real number α.

Ex. 1.4: Eigenvectors and Eigenvalues

(The notation $\alpha \in \mathbb{R}$ in (5) means: α is an element of the real numbers $\mathbb{R}$.)

For $\lambda_2 = 2$, we add the first row to the third row.

$$\left(\begin{array}{ccc|c} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then we multiply the first row by 2 and multiply the second row by (-1)

$$\Leftrightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Leftrightarrow \quad \mathbf{x}_2 = \beta \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \beta \in \mathbb{R}.$$

In the last step we have used that the linear system provides the equations $x - z = 0$ and $y = 0$ if we denote $\mathbf{x}_1 = (x, y, z)'$. Hence $y = 0$ and $x = z = \beta$ for any choice of the real number β.

Ex. 1.4: Eigenvectors and Eigenvalues

For $\lambda_3 = 1$, we subtract the first row from the third row.

$$\left(\begin{array}{ccc|c} -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -2 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \end{array}\right) \quad \Leftrightarrow \quad \left(\begin{array}{ccc|c} -\frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then we multiply the first row by (-2) and multiply the second row by $(-1/2)$ and obtain

$$\Leftrightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \Leftrightarrow \quad \mathbf{x}_3 = \gamma \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \gamma \in \mathbb{R}.$$

In the last step we have used that the linear system provides the equations $x + z = 0$ and $y = 0$ if we denote $\mathbf{x}_3 = (x, y, z)'$. Hence $y = 0$, $z = -x$ and $x = \gamma$ for any real number γ.

Ex. 1.4: Eigenvectors and Eigenvalues

We summarize our results so far:

$$\lambda_1 = 3 \qquad \text{has the eigenvectors} \qquad \mathbf{x}_1 = \begin{pmatrix} 0 \\ \alpha \\ 0 \end{pmatrix},$$

$$\lambda_2 = 2 \qquad \text{has the eigenvectors} \qquad \mathbf{x}_2 = \begin{pmatrix} \beta \\ 0 \\ \beta \end{pmatrix},$$

$$\lambda_3 = 1 \qquad \text{has the eigenvectors} \qquad \mathbf{x}_3 = \begin{pmatrix} \gamma \\ 0 \\ -\gamma \end{pmatrix},$$

where the real numbers α, β, γ can have *any value apart from zero*. (Eigenvectors must be different from the zero vector; hence we must exclude $\alpha = 0$, $\beta = 0$ and $\gamma = 0$.)

Solution of Part 2: Since $\mathbf{A}$ is *symmetric*, its *eigenvectors to different eigenvalues are orthogonal*. Thus we obtain a suitable orthogonal matrix $\mathbf{S}$ by choosing *normalized eigenvectors* (i.e. eigenvectors with length 1).

From the results in the previous part of this question, the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

are normalized eigenvectors to the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 1$, respectively, and they are orthogonal to each other. (*Note*: To get a normalized eigenvector, divide the eigenvector by its length.)

Thus we choose the *orthogonal matrix* to be

$$\mathbf{S} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \mathbf{S}' = \mathbf{S}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Ex. 1.4: Eigenvectors and Eigenvalues

To confirm that we have correctly chosen an *orthogonal matrix* $\mathbf{S}$, we execute the matrix multiplications $\mathbf{S}'\,\mathbf{S}$ and $\mathbf{S}\,\mathbf{S}'$.

$$\mathbf{S}'\,\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{S}\,\mathbf{S}' = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This shows that $\mathbf{S}'\,\mathbf{S} = \mathbf{S}\,\mathbf{S}' = \mathbf{I}$ and hence $\mathbf{S}' = \mathbf{S}^{-1}$, i.e. our $\mathbf{S}$ is an orthogonal matrix.

Ex. 1.4: Eigenvectors and Eigenvalues

From executing the matrix multiplications, we find

$$\begin{aligned}
\mathbf{S}'\,\mathbf{A}\,\mathbf{S} &= \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 3 & 0 \\ \frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} & \frac{1}{\sqrt{2}} \\ 3 & 0 & 0 \\ 0 & \sqrt{2} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\
&= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

as desired.

Ex. 1.5: Mean, Variance, Covariance and Correlation

Consider the random variables $X =$ *mark of students in percentage* and $Y =$ *age of the student*. In a sample of 3 students we found the values

$$x_1 = 80,\ x_2 = 90,\ x_3 = 70 \qquad \text{and} \qquad y_1 = 24,\ y_2 = 23,\ y_3 = 22$$

for X and Y, respectively. *Estimate* the *covariance* and the *correlation coefficient* of X and Y from the sample.

<u>Solution:</u> From the examples on the lecture slides, we already know that the *mean of* X is $\overline{x} = 80$ and that the *empirical standard deviation of* X is $s_X = 10$.

$$\overline{y} = \frac{1}{3}\left(y_1 + y_2 + y_3\right) = \frac{1}{3}(24 + 23 + 22) = \frac{69}{3} = 23$$

$$s_Y^2 = \frac{1}{3-1}\left[(y_1 - \overline{y})^2 + (y_2 - \overline{y})^2 + (y_3 - \overline{y})^2\right]$$

$$= \frac{1}{2}\left[(24-23)^2 + (23-23)^2 + (22-23)^2\right] = \frac{1}{2}\left[1^2 + (-1)^2\right] = 1$$

Ex. 1.5: Mean, Variance, Covariance and Correlation

Hence we find that the *mean of Y* is $\overline{y} = 23$ and that the *empirical standard deviation of Y* is $s_Y = \sqrt{1} = 1$.

Next we compute the *empirical covariance* of X and Y:

$$\begin{aligned}
&\widehat{\text{Cov}}(X, Y) \\
&= \frac{1}{3-1}\Big[(x_1 - \overline{x})(y_1 - \overline{y}) + (x_2 - \overline{x})(y_2 - \overline{y})^2 + (x_3 - \overline{x})(y_3 - \overline{y})^2\Big] \\
&= \frac{1}{2}\Big[(80-80)\cdot(24-23) + (90-80)\cdot(23-23) + (70-80)\cdot(22-23)\Big] \\
&= \frac{1}{2}\Big[0\cdot 1 + 10\cdot 0 + (-10)\cdot(-1)\Big] = \frac{10}{2} = 5.
\end{aligned}$$

The *empirical correlation coefficient* is given by

$$\widehat{\varrho}(X, Y) = \frac{\widehat{\text{Cov}}(X, Y)}{s_X \cdot s_Y} = \frac{5}{10\cdot 1} = \frac{1}{2}.$$

Ex. 1.6: Formal Manipulations of Expectation Values

Let X, Y and W be random variables with expectation values $\mu_X = \mathrm{E}(X)$, $\mu_Y = E(Y)$ and $\mu_W = E(W)$ and standard deviations σ_X, σ_Y and σ_W, respectively. Let a, b and c be real numbers. Use

$$\mathrm{E}(a \cdot X + b \cdot Y) = a \cdot \mathrm{E}(X) + b \cdot \mathrm{E}(Y). \tag{6}$$

to *verify the following relations*:

$$\begin{aligned} \mathrm{E}\Big(\big[a \cdot (X - \mu_X) + b \cdot (Y - \mu_Y) \big] \cdot \big[c \cdot (W - \mu_W) \big] \Big) \\ = a \cdot c \cdot \mathrm{Cov}(X, W) + b \cdot c \cdot \mathrm{Cov}(Y, W), \end{aligned}$$

$$\mathrm{Var}(a \cdot X) = a^2 \cdot \mathrm{Var}(X).$$

Solution: We start by determining the various terms from executing the multiplication of the two terms of which we take the expectation value.

Ex. 1.6: Formal Manipulations of Expectation Values

$$\begin{aligned}
&\mathrm{E}\Big(\big[a\cdot(X-\mu_X)+b\cdot(Y-\mu_Y)\big]\cdot\big[c\cdot(W-\mu_W)\big]\Big)\\
&= \mathrm{E}\Big(a\cdot(X-\mu_X)\cdot c\cdot(W-\mu_W)+b\cdot(Y-\mu_Y)\cdot c\cdot(W-\mu_W)\Big)\\
&= \mathrm{E}\Big((a\cdot c)\cdot(X-\mu_X)\cdot(W-\mu_W)+(b\cdot c)\cdot(Y-\mu_Y)\cdot(W-\mu_W)\Big)\\
&= (a\cdot c)\cdot\mathrm{E}\Big((X-\mu_X)\cdot(W-\mu_W)\Big)+(b\cdot c)\cdot\mathrm{E}\Big((Y-\mu_Y)\cdot(W-\mu_W)\Big)\\
&= (a\cdot c)\cdot\mathrm{Cov}(X,W)+(b\cdot c)\cdot\mathrm{Cov}(Y,W),
\end{aligned}$$

where we have used (6) in the 4th step. We note that it was essential that we kept the centered variables $(X-\mu_X)$, $(Y-\mu_Y)$ and $(W-\mu_W)$.

To verify $\mathrm{Var}(a\cdot X)=a^2\cdot\mathrm{Var}(X)$, we express $\mathrm{Var}(a\cdot X)$ as an expectation value: Using that (from (6) with $b=0$) $\mathrm{E}(a\cdot X)=a\cdot\mathrm{E}(X)$, we have

$$\begin{aligned}
\mathrm{Var}(a\cdot X) &= \mathrm{E}\big([a\cdot X-\mathrm{E}(a\cdot X)]^2\big)=\mathrm{E}\big([a\cdot X-a\cdot\mathrm{E}(X)]^2\big)\\
&= \mathrm{E}\big(a^2\cdot[X-\mathrm{E}(X)]^2\big)=a^2\cdot\mathrm{E}\big([X-\mathrm{E}(X)]^2\big)=a^2\cdot\mathrm{Var}(X).
\end{aligned}$$

Structural Equation Modeling

Solutions to Topic 2: Factor Analysis

Dr. Kerstin Hesse

Email: kerstin.hesse@hhl.de; *Phone:* +49 (0)341 9851-820; *Office:* HHL Main Building, Room 115A

HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany

Doctoral Program at HHL, June 1-2, 2012

Ex. 2.1: Standardized Data in Example & Toy Exercise

Given the rating for vitamins ($= X_1$), rating for calories ($= X_2$), rating for shelf live date ($= X_3$) and rating for price ($= X_4$) for 5 types of cereal in the following table, compute the data for the corresponding *standardized variables* $Z_1, \ldots, Z_4$ and write down the *standardized data matrix*:

Cereal	X_1 (Vitamins)	X_2 (Calories)	X_3 (Shelf Live)	X_4 (Price)
e_1	4	2	3	3
e_2	2	4	3	3
e_3	3	3	3	3
e_4	3	3	2	4
e_5	3	3	4	2

Ex. 2.1: Standardized Data in Example & Toy Exercise

Solution: For each random variable X_j, we first compute the *mean* $\overline{x_j}$

$$\overline{x_1} = \frac{1}{5} \cdot \big(4+2+3+3+3\big) = \frac{15}{3} = 3,$$

$$\overline{x_2} = \frac{1}{5} \cdot \big(2+4+3+3+3\big) = \frac{15}{3} = 3,$$

$$\overline{x_3} = \frac{1}{5} \cdot \big(3+3+3+2+4\big) = \frac{15}{3} = 3,$$

$$\overline{x_4} = \frac{1}{5} \cdot \big(3+3+3+4+2\big) = \frac{15}{3} = 3,$$

and the *empirical variance* s_j^2 and *empirical standard deviation* s_j

$$s_1^2 = \frac{1}{4} \cdot \big(1^2 + (-1)^2 + 0 + 0 + 0\big) = \frac{2}{4} = \frac{1}{2} \qquad \Rightarrow \qquad s_1 = \frac{1}{\sqrt{2}},$$

$$s_2^2 = \frac{1}{4} \cdot \big((-1)^2 + 1^2 + 0 + 0 + 0\big) = \frac{2}{4} = \frac{1}{2} \qquad \Rightarrow \qquad s_2 = \frac{1}{\sqrt{2}},$$

$$s_3^2 = \frac{1}{4} \cdot \big(0 + 0 + 0 + (-1)^2 + 1^2\big) = \frac{2}{4} = \frac{1}{2} \qquad \Rightarrow \qquad s_3 = \frac{1}{\sqrt{2}},$$

$$s_4^2 = \frac{1}{4} \cdot \big(0 + 0 + 0 + 1^2 + (-1)^2\big) = \frac{2}{4} = \frac{1}{2} \qquad \Rightarrow \qquad s_4 = \frac{1}{\sqrt{2}}.$$

Ex. 2.1: Standardized Data in Example & Toy Exercise

The *standardized data* for the variable X_j and the cereal e_i is given by

$$z_{ij} = \frac{x_{ij} - \overline{x_j}}{s_j} = \frac{x_{ij} - 3}{1/\sqrt{2}} = \sqrt{2}\cdot(x_{ij} - 3), \text{ where } x_{ij} = \text{value of } X_j \text{ for cereal } e_i$$

Thus we find the *standardized data matrix*:

$$\mathbf{Z} = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 & 0 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \begin{matrix} \leftarrow \text{ standardized data for cereal } e_1 \\ \leftarrow \text{ standardized data for cereal } e_2 \\ \leftarrow \text{ standardized data for cereal } e_3 \\ \leftarrow \text{ standardized data for cereal } e_4 \\ \leftarrow \text{ standardized data for cereal } e_5 \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow \\ Z_1 & Z_2 & Z_3 & Z_4 \end{matrix}$$

where Z_j is the *standardized variable* the corresponds to X_j.

Ex. 2.2: Some Model Equations in the Toy Exercise

Write down the *model equations* for each random variable X_j for cereals e_1 and e_2. Inspect the model equations:

- What are the *unknowns*?
- Compare the model equations with the equations in *(multiple) regression*. Where lies the *difference*?

Solution We start by inspecting an individual equation: For cereal e_1 and random variable Z_1 (standardized variable corresponding to X_1) we have the model equation:

$$\sqrt{2} = a_{1,1} \cdot f_{1,1} + a_{1,2} \cdot f_{1,2} + \ldots + a_{1,p} \cdot f_{1,p} + u_{1,1}.$$

- The factor loadings a_{jk} depend on the random variable X_j and the factors F_k but *not* on the different types of cereal.
- The values f_{ik} of the factors F_k depend on the different types of cereal e_i but are the same for all random variables X_j.
- The unique factors u_{ik} depend on the random variable X_k and on the type of cereal e_i.

Ex. 2.2: Some Model Equations in the Toy Exercise

r. var.	model equations for e_1	model equations for e_2
Z_1	$\sqrt{2} = a_{1,1} \cdot f_{1,1} + a_{1,2} \cdot f_{1,2} + \ldots + a_{1,p} \cdot f_{1,p} + u_{1,1}$	$-\sqrt{2} = a_{1,1} \cdot f_{2,1} + a_{1,2} \cdot f_{2,2} + \ldots + a_{1,p} \cdot f_{2,p} + u_{2,1}$
Z_2	$-\sqrt{2} = a_{2,1} \cdot f_{1,1} + a_{2,2} \cdot f_{1,2} + \ldots + a_{2,p} \cdot f_{1,p} + u_{1,2}$	$\sqrt{2} = a_{2,1} \cdot f_{2,1} + a_{2,2} \cdot f_{2,2} + \ldots + a_{2,p} \cdot f_{2,p} + u_{2,2}$
Z_3	$0 = a_{3,1} \cdot f_{1,1} + a_{3,2} \cdot f_{1,2} + \ldots + a_{3,p} \cdot f_{1,p} + u_{1,3}$	$0 = a_{3,1} \cdot f_{2,1} + a_{3,2} \cdot f_{2,2} + \ldots + a_{3,p} \cdot f_{2,p} + u_{2,3}$
Z_4	$0 = a_{4,1} \cdot f_{1,1} + a_{4,2} \cdot f_{1,2} + \ldots + a_{4,p} \cdot f_{1,p} + u_{1,4}$	$0 = a_{4,1} \cdot f_{2,1} + a_{4,2} \cdot f_{2,2} + \ldots + a_{4,p} \cdot f_{2,p} + u_{2,4}$

We note that in each equation the factor loadings (coefficients) a_{jk} and the values of the factors f_{ik} are the *unknowns*.

Ex. 2.2: Some Model Equations in the Toy Exercise

The model equation for cereal e_1 and random variable X_1

$$\sqrt{2} = a_{1,1} \cdot f_{1,1} + a_{1,2} \cdot f_{1,2} + \ldots + a_{1,p} \cdot f_{1,p} + u_{1,1}.$$

looks like the equation of a *(multivariate) regression*.

However, in *regression* we would also know *values for the factors*, but these are *unknown*!

Ex. 2.3: Correlation Matrix

Compute the *correlation matrix* for our toy example.

Solution: Using the standardized data matrix from Ex. 2.1 we have

$$\mathbf{R} = \frac{1}{5-1}\mathbf{Z}'\,\mathbf{Z}$$

$$= \frac{1}{4}\begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 & 0 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 4 & -4 & 0 & 0 \\ -4 & 4 & 0 & 0 \\ 0 & 0 & 4 & -4 \\ 0 & 0 & -4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Ex. 2.4: Communalities and Reduced Correlation Matrix

For our toy example, *estimate the communalities* with method 1 (see page 57 of the lecture slides) and *estimate* $\mathbf{R}_h$ for our toy example.

<u>Solution:</u> In Ex. 2.3 we found the *correlation matrix*

$$\mathbf{R} = (r_{i,k}) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

and from method 1 we get the following *estimates of the communalities*:

$$\widehat{h_1}^2 = \max_{k \neq 1} |r_{1,k}| = \max\{0, |-1|\} = 1,$$
$$\widehat{h_2}^2 = \max_{k \neq 2} |r_{2,k}| = \max\{0, |-1|\} = 1,$$
$$\widehat{h_3}^2 = \max_{k \neq 3} |r_{3,k}| = \max\{0, |-1|\} = 1,$$
$$\widehat{h_4}^2 = \max_{k \neq 4} |r_{4,k}| = \max\{0, |-1|\} = 1.$$

Ex. 2.4: Communalities and Reduced Correlation Matrix

To estimate the reduced correlation matrix $\mathbf{R}_h = \mathbf{R} - \mathbf{\Psi}$, we need to replace the jth diagonal entry r_{jj} of $\mathbf{R}$ by the estimate of communality $\widehat{h_j}^2$. Here we find that

$$r_{jj} = \widehat{h_j}^2 = 1 \qquad \text{for } j = 1, 2, \ldots, 4.$$

Hence

$$\widehat{\mathbf{R}_h} = \mathbf{R} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix},$$

i.e. the estimated reduced correlation matrix $\widehat{\mathbf{R}_h}$ is *identical* to the correlation matrix $\mathbf{R}$.

Ex. 2.5: Estimating the Factor Loading Matrix with PCA

Estimate the *factor loading matrix* **A** for our toy example using the *Kaiser criterion*. Write down the explicit model equations and *interpret* them.

Solution: *Step 1*: We start by computing the *eigenvalues* of $\mathbf{R} = \mathbf{R}_h$:

$$\det(\lambda\,\mathbf{I} - \mathbf{R}) = \begin{pmatrix} \lambda-1 & 1 & 0 & 0 \\ 1 & \lambda-1 & 0 & 0 \\ 0 & 0 & \lambda-1 & 1 \\ 0 & 0 & 1 & \lambda-1 \end{pmatrix}$$

$$= (\lambda-1)\det\begin{pmatrix} \lambda-1 & 0 & 0 \\ 0 & \lambda-1 & 1 \\ 0 & 1 & \lambda-1 \end{pmatrix} - \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda-1 & 1 \\ 0 & 1 & \lambda-1 \end{pmatrix}$$

$$= (\lambda-1)\cdot\left[(\lambda-1)^3 - (\lambda-1)\right] - \left[(\lambda-1)^2 - 1\right],$$

where we have expanded the determinant with respect to the first row and then used the formula for the determinants of 3×3 matrices.

Ex. 2.5: Estimation of the Factor Loading Matrix

We simplify, and use the binomial formulas $a^2 - 2\cdot a\cdot b + b^2 = (a-b)^2$ and $c^2 - d^2 = (c-d)\cdot(c+d)$.

$$\begin{aligned}
\det(\lambda\,\mathbf{I} - \mathbf{R}) &= (\lambda-1)\cdot\left[(\lambda-1)^3 - (\lambda-1)\right] - \left[(\lambda-1)^2 - 1\right] \\
&= \underbrace{\left[(\lambda-1)^2\right]^2}_{=a^2} - \underbrace{2\,(\lambda-1)^2}_{=-2\cdot a\cdot b} + \underbrace{1}_{=b^2} = \Big[\underbrace{(\lambda-1)^2}_{=a} - \underbrace{1}_{=b}\Big]^2 \\
&= \Big[\underbrace{(\lambda-1)^2}_{=c^2} - \underbrace{1}_{=d^2}\Big]^2 = \Big[\underbrace{(\lambda-1-1)}_{=c-d}\cdot\underbrace{(\lambda-1+1)}_{=c+d}\Big]^2 \\
&= \left[(\lambda-2)\cdot\lambda\right]^2 = (\lambda-2)^2\cdot\lambda^2
\end{aligned}$$

Thus we find the *eigenvalues*

$$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \lambda_3 = 0, \quad \lambda_4 = 0.$$

Next we compute the *corresponding eigenvectors* by solving the linear system $(\lambda_j\,\mathbf{I} - \mathbf{R})\,\mathbf{b}_j = \mathbf{0}$ for each eigenvalue λ_j.

Ex. 2.5: Estimation of the Factor Loading Matrix

For $\lambda_1 = \lambda_2 = 2$ we have to solve:

$$(2\,\mathbf{I} - \mathbf{R}\,|\,\mathbf{0}) = \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right) \quad \Leftrightarrow \quad \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

In the first step, we have subtracted the 1st row from the 2nd row, and we have subtracted the 3rd row from the 4th row.

Thus we obtain for the eigenvectors $\mathbf{b} = (w, x, y, z)'$ the equations

$$\big(\; w + x = 0 \quad \text{and} \quad y + z = 0 \;\big) \quad \Leftrightarrow \quad \big(\; x = -w \quad \text{and} \quad z = -y \;\big)$$

From these equations, two *normalized orthogonal eigenvectors* for $\lambda_1 = \lambda_2 = 2$ are

$$\mathbf{b}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Ex. 2.5: Estimation of the Factor Loading Matrix

For $\lambda_3 = \lambda_4 = 0$ we have to solve:

$$(0\,\mathbf{I} - \mathbf{R}\,|\,\mathbf{0}) = \left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right) \Leftrightarrow \left(\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

In the first step, we have added the 1st row from the 2nd row, and we have added the 3rd row from the 4th row.

Thus we obtain for the eigenvectors $\mathbf{b} = (w, x, y, z)'$ the equations

$$\big(-w + x = 0 \quad \text{and} \quad -y + z = 0 \big) \quad \Leftrightarrow \quad \big(x = w \quad \text{and} \quad z = y \big)$$

From these equations, two *normalized orthogonal eigenvectors* for $\lambda_3 = \lambda_4 = 0$ are

$$\mathbf{b}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b}_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Ex. 2.5: Estimation of the Factor Loading Matrix

Step 2: We have found only *2 positive eigenvalues* $\lambda_1 = \lambda_2 = 2$ with two *corresponding orthogonal eigenvectors*

$$\mathbf{b}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

Thus we initially choose

$$\mathbf{A} = \left(\sqrt{\lambda_1}\,\mathbf{b}_1, \sqrt{\lambda_2}\,\mathbf{b}_2\right) = \left(\sqrt{2}\,\mathbf{b}_1, \sqrt{2}\,\mathbf{b}_2\right) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Step 3: The *Kaiser criterion* suggests to use only those eigenvalues λ_j (and the corresponding eigenvectors $\mathbf{b}_j$) that satisfy $\lambda_j > 1$.

For our example we have $\lambda_1 = \lambda_2 = 2 > 1$, and hence we keep our initial choice of $\mathbf{A}$, and we have found $p = 2$ *factors*.

Ex. 2.5: Estimation of the Factor Loading Matrix

Out of interest, we test how well $\mathbf{A}\,\mathbf{A}'$ reproduces the matrix $\mathbf{R}$:

$$\mathbf{A}\,\mathbf{A}' = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{=\ \mathbf{R}}$$

We note that we are here in the *unusual* situation that

$$\mathbf{R} = \mathbf{A}\,\mathbf{A}' + \mathbf{\Psi} \qquad \text{with} \qquad \mathbf{\Psi} = \mathbf{0}.$$

As $\mathbf{\Psi}$ is the model covariance matrix of the unique factors, $\mathbf{\Psi} = \mathbf{0}$ tells us that $\psi_{jj} = 0$ (i.e. $\mathrm{Var}(U_j)$ is estimated to be zero) for $j = 1, 2, \ldots, 4$. Since by assumption $\mathrm{E}(U_j) = 0$, based on our sampled data we expect $U_j = 0$ for $j = 1, 2, \ldots, 4$.

Thus, based on our sample, our factor analysis model with the two factors appears to be an *exact model without model errors*.

Ex. 2.5: Estimation of the Factor Loading Matrix

Explicit Model Equations: From

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}$$

we have: $a_{1,1} = 1$, $a_{1,2} = 0$ (for Z_1); $a_{2,1} = -1$, $a_{2,2} = 0$ (for Z_2); $a_{3,1} = 0$, $a_{3,2} = 1$ (for Z_3); and $a_{4,1} = 0$, $a_{4,2} = -1$ (for Z_4).

Thus the *model equations* are given by:

$$\begin{aligned} Z_1 &= a_{1,1} \cdot F_1 + a_{1,2} \cdot F_2 + U_1 = F_1 + U_1 = F_1, \\ Z_2 &= a_{2,1} \cdot F_1 + a_{2,2} \cdot F_2 + U_1 = -F_1 + U_2 = -F_1, \\ Z_3 &= a_{3,1} \cdot F_1 + a_{3,2} \cdot F_2 + U_1 = F_2 + U_3 = F_2, \\ Z_4 &= a_{4,1} \cdot F_1 + a_{4,2} \cdot F_2 + U_1 = -F_2 + U_4 = -F_2, \end{aligned}$$

where, in the last step, we have used that our factor analysis model *appears to be exact* (no error terms U_j required; see last page).

Ex. 2.5: Estimation of the Factor Loading Matrix

Interpretation of the Model Equations and the Factors:

$$\begin{aligned} Z_1 &= F_1 + U_1 = F_1, \\ Z_2 &= -F_1 + U_2 = -F_1, \\ Z_3 &= F_2 + U_3 = F_2, \\ Z_4 &= -F_2 + U_4 = -F_2, \end{aligned}$$

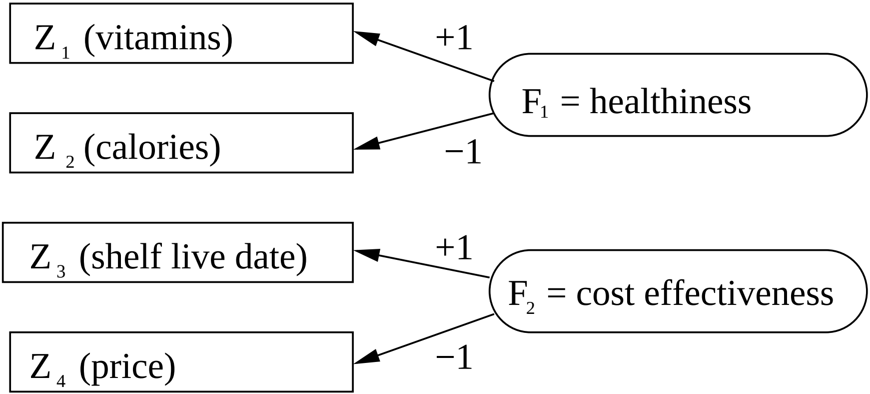
We observe that:

- F_1 is *positively correlated* to $X_1 =$ rating for vitamins and *negatively correlated* to $X_2 =$ rating for calories. F_1 is *uncorrelated* to $X_3 =$ rating for shelf life date and $X_4 =$ rating for price.
- F_2 is *positively correlated* to $X_3 =$ rating for shelf life date and *negatively correlated* to $X_4 =$ rating for price. F_2 is *uncorrelated* to $X_1 =$ rating for vitamins and $X_2 =$ rating for calories.

Thus we may *interpret F_1 as healthiness* and F_2 *as cost effectiveness*.

Ex. 2.5: Estimation of the Factor Loading Matrix

The diagram below describes our *factor analytic model*:



Ex. 2.6: Factor Values

Compute the *factor values* for our toy example.

Solution: From the *least squares equations* we have to compute

$$\mathbf{F}' = (\mathbf{A}'\,\mathbf{A})^{-1}\mathbf{A}'\,\mathbf{Z}'.$$

We start by computing $\mathbf{A}'\,\mathbf{A}$ and its inverse matrix $(\mathbf{A}'\,\mathbf{A})^{-1}$

$$\mathbf{A}'\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$(\mathbf{A}'\mathbf{A})^{-1} = = \frac{1}{4} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Ex. 2.6: Factor Values

Next we compute $\mathbf{F}'$ from $\mathbf{F}' = (\mathbf{A}'\,\mathbf{A})^{-1}\mathbf{A}'\mathbf{Z}'$.

$$\begin{aligned}
\mathbf{F}' &= (\mathbf{A}'\,\mathbf{A})^{-1}\mathbf{A}'\,\mathbf{Z}' \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\sqrt{2} & 2\sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} \end{pmatrix}
\end{aligned}$$

Ex. 2.6: Factor Values

Taking the transpose of $\mathbf{F}'$, the matrix $\mathbf{F}$ of the factor values is given by

$$\mathbf{F}(f_{ij}) = \begin{pmatrix} \sqrt{2} & 0 \\ -\sqrt{2} & 0 \\ 0 & 0 \\ 0 & -\sqrt{2} \\ 0 & \sqrt{2} \end{pmatrix}$$

We find that the *factor values* in our example are:

- $f_{1,1} = \sqrt{2}$ and $f_{1,2} = 0$ for cereal e_1
- $f_{2,1} = -\sqrt{2}$ and $f_{2,2} = 0$ for cereal e_2
- $f_{3,1} = 0$ and $f_{3,2} = 0$ for cereal e_3
- $f_{4,1} = 0$ and $f_{4,2} = -\sqrt{2}$ for cereal e_4
- $f_{5,1} = 0$ and $f_{5,2} = \sqrt{2}$ for cereal e_5

Ex. 2.7: Interpretation

Interpret the factor values for our example.

Solution: The factor values $f_{1,1} = \sqrt{2}$, $f_{1,2} = 0$ for cereal e_1 indicate an *above average healthiness* and an average cost effectiveness.

The factor values $f_{2,1} = -\sqrt{2}$, $f_{2,2} = 0$ for cereal e_2 indicate a *below average healthiness* and an average cost effectiveness.

The factor values $f_{3,1} = 0$, $f_{3,2} = 0$ for cereal e_3 indicate an average healthiness and an average cost effectiveness.

The factor values $f_{4,1} = 0$, $f_{4,2} = -\sqrt{2}$ for cereal e_4 indicate an average healthiness and a *below average cost effectiveness*.

The factor values $f_{5,1} = 0$, $f_{5,2} = \sqrt{2}$ for cereal e_5 indicate an average healthiness and an *above average cost effectiveness*.

These interpretations *agree with the ratings* given as data in our example.

Structural Equation Modeling

Solutions to Topic 3: Introduction to Structural Equation Modeling (SEM)

Dr. Kerstin Hesse

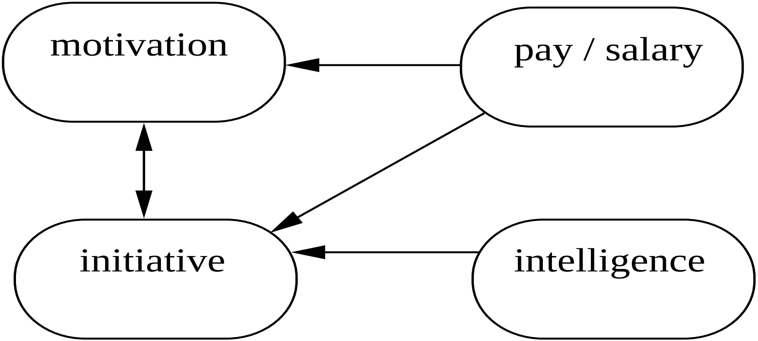
Email: kerstin.hesse@hhl.de; *Phone:* +49 (0)341 9851-820; *Office:* HHL Main Building, Room 115A

HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany

Doctoral Program at HHL, June 1-2, 2012

Ex. 3.1: Setting up the Structural (Inner) Model

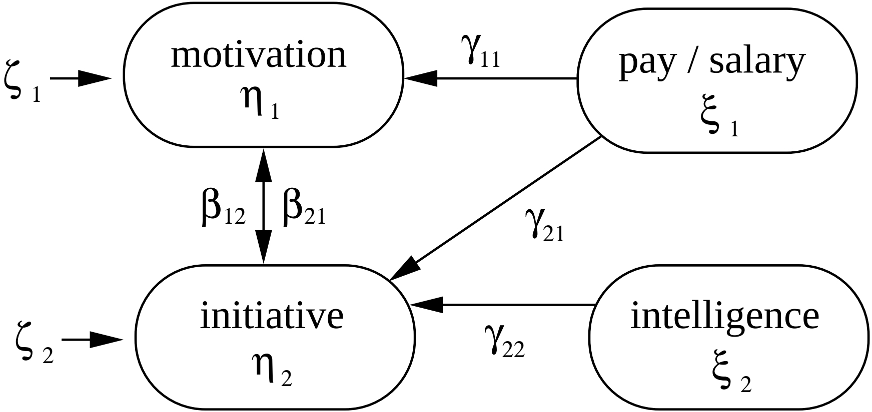
A model for the *work of a software programmer on an non-pay-scale salary* is shown in the diagram below. *Indicate the various latent variables and the coefficients and error terms in the diagram*, using the rules explained on pages 71–72 of the lecture slides. For consistency, number any exogenous (or endogenous) latent variables from top to bottom. Finally write down the *equations for the structural (inner) model*.



Ex. 3.1: Setting up the Structural (Inner) Model

Solution: Exogenous latent variables: ξ_1 = pay/salary, ξ_2 = intelligence. Endogenous latent variables: η_1 = motivation, η_2 = initiative.

We note that we have a *two-way relationship* between η_1 = motivation and η_2 = initiative; they influence *each other*.



Ex. 3.1: Setting up the Structural (Inner) Model

The *structural (inner) model* is given by

$$\begin{aligned} \eta_1 &= \beta_{1,2}\,\eta_2 + \gamma_{1,1}\,\xi_1 + \zeta_1 \\ \eta_2 &= \beta_{2,1}\,\eta_1 + \gamma_{2,1}\,\xi_1 + \gamma_{2,2}\,\xi_2 + \zeta_2 \end{aligned}$$

or equivalently in *matrix notation*

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \gamma_{1,1} & 0 \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},$$

i.e. $\boldsymbol{\eta} = \mathbf{B}\,\boldsymbol{\eta} + \boldsymbol{\Gamma}\,\boldsymbol{\xi} + \boldsymbol{\zeta}$ with

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \boldsymbol{\zeta} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix}, \qquad \boldsymbol{\Gamma} = \begin{pmatrix} \gamma_{1,1} & 0 \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix}.$$

Ex. 3.2: Reduced Model

Write down the *reduced model* for the structural (inner) model from Ex. 3.1.

Solution: In Ex. 3.1 we found the linear system

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \gamma_{1,1} & 0 \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

We subtract the first term on the right-hand side on both sides:

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} & 0 \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}. \tag{7}$$

Next we transform the left-hand side in (7)

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} - \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

Ex. 3.2: Reduced Model

$$= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix} \right] \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \beta_{1,2} \\ \beta_{2,1} & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

We substitute the result back into (7) and get

$$\underbrace{\begin{pmatrix} 1 & \beta_{1,2} \\ \beta_{2,1} & 1 \end{pmatrix}}_{= \mathbf{B}^*} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} \gamma_{1,1} & 0 \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

With

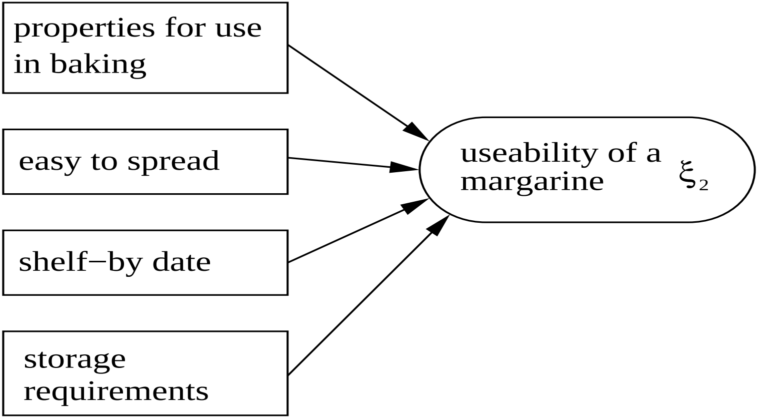
$$(\mathbf{B}^*)^{-1} = \frac{1}{1 - \beta_{2,1}\,\beta_{1,2}} \begin{pmatrix} 1 & -\beta_{1,2} \\ -\beta_{2,1} & 1 \end{pmatrix}$$

we now obtain the reduced model

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = (\mathbf{B}^*)^{-1} \begin{pmatrix} \gamma_{1,1} & 0 \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + (\mathbf{B}^*)^{-1} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

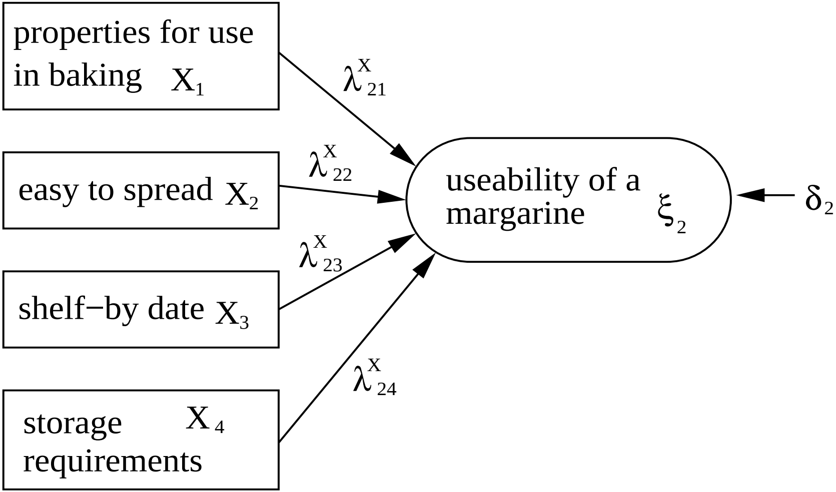
Ex. 3.3: Formative Measurement Model

Starting the numbering of the measurement variables at the top, indicate the *measurement variables, error terms* and *coefficients* in the following diagram of a *formative measurement model*. Then write down the *regression equation* for the exogenous latent variable ξ_2.



Ex. 3.3: Formative Measurement Model

Solution: First we complete the diagram.



Ex. 3.3: Formative Measurement Model

We note that we call the measurement variables X_i because the latent variable is an *exogenous* latent variable (indicated by its name ξ_2).

(The numbering of the measurement variables is of course arbitrary; we could equally well have started from the bottom rather than from the top. However, a change in the numbering of the measurement variables will also result in a change of the indices of the path coefficients.)

Likewise we call the error term δ_2 since the latent variable ξ_2 is *exogenous*.

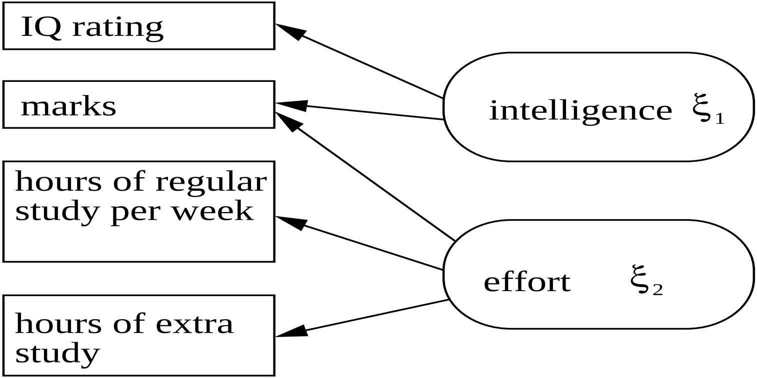
Further we note that the coefficient between X_i and ξ_2 is λ^X_{2i} because the arrow points from X_i to ξ_2. (*Notation*: first index of the coefficient = index of the variable that the arrow is pointing to; second index = index of the variable that the arrow originates at.)

Regression equation for the measurement model of the latent variable:

$$\xi_2 = \lambda^X_{2,1}\,(X_1 - \mu_{X_1}) + \lambda^X_{2,2}\,(X_2 - \mu_{X_2}) + \lambda^X_{2,3}\,(X_3 - \mu_{X_3}) + \lambda^X_{2,4}\,(X_4 - \mu_{X_4}) + \delta_2$$

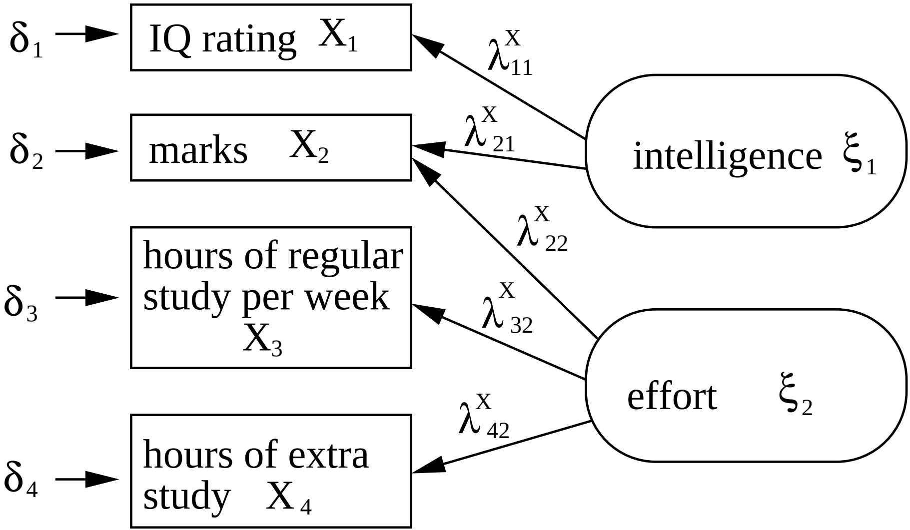
Ex. 3.4: Reflective Measurement Model

The diagram below shows part of a structural equation model for the academic success of students. Numbering the measurement variables from the top to the bottom, complete the diagram of the reflective measurement model by indicating the *variables*, *error terms* and *coefficients*. Then write down the *factor analytic equations* for the measurement variables.



Ex. 3.4: Reflective Measurement Model

Solution: First we complete the diagram.



Ex. 3.4: Reflective Measurement Model

We note that we call the measurement variables X_i and call their error terms δ_i because the latent variables are *exogenous* latent variables (as indicated by their names ξ_1 and ξ_2).

Further we note that the coefficient between X_i and ξ_j is λ_{ij}^X because the arrow points from ξ_j to X_i. (*Notation*: first index of the coefficient = index of the variable that the arrow is pointing to; second index = index of the variable that the arrow originates at.)

Factor analytical equations for the measurement model of the latent variables:

$$\begin{aligned}
X_1 - \mu_{X_1} &= \lambda_{1,1}^X \xi_1 + \delta_1 \\
X_2 - \mu_{X_2} &= \lambda_{2,1}^X \xi_1 + \lambda_{2,2}^X \xi_2 + \delta_2 \\
X_3 - \mu_{X_3} &= \lambda_{3,2}^X \xi_2 + \delta_3 \\
X_4 - \mu_{X_4} &= \lambda_{4,2}^X \xi_2 + \delta_4
\end{aligned}$$

Structural Equation Modeling

Solutions to Topic 4:

LISREL (Linear Structural Relationships)

Dr. Kerstin Hesse

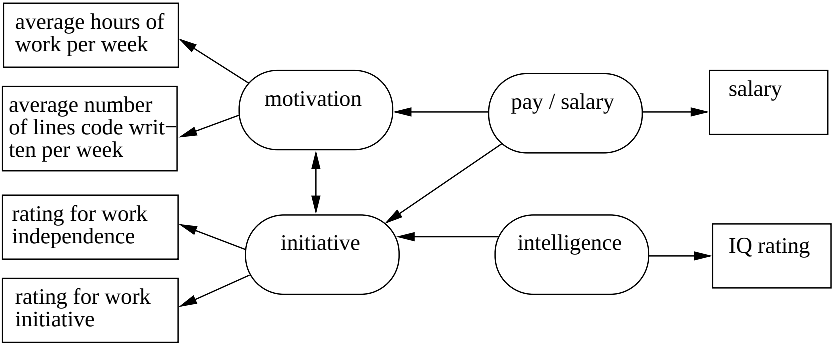
Email: kerstin.hesse@hhl.de; *Phone:* +49 (0)341 9851-820; *Office:* HHL Main Building, Room 115A

HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany

Doctoral Program at HHL, June 1-2, 2012

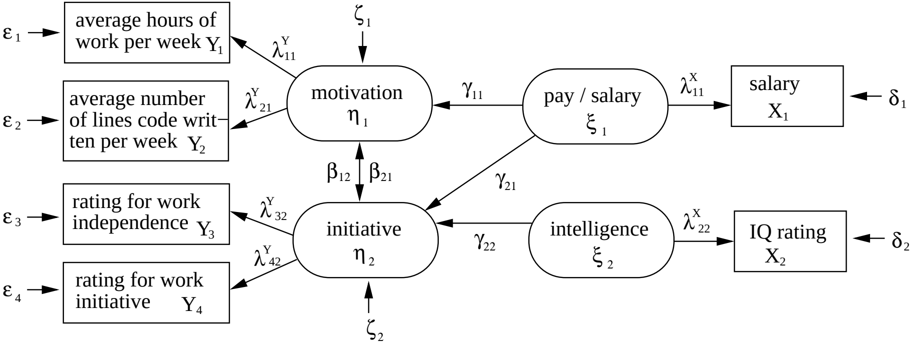
Ex. 4.1: Structural Equation Model with LISREL

The structural model for the *work of a software programmer on a non-pay-scale salary* (see Ex. 3.1) has now been equipped with the *reflective measurement models* for the latent variables shown below. *Indicate all variables, errors and coefficients in the diagram* and *write down the equations of the measurement models.* The ratings (apart from the IQ one) have been provided by the programmer's superior.



Ex. 4.1: Structural Equation Model with LISREL

Solution: First we complete the diagram by indicating all variables, errors and coefficients.



Structural (inner) model (from Ex. 3.1):

$$\eta_1 = \beta_{1,2}\,\eta_2 + \gamma_{1,1}\,\xi_1 + \zeta_1$$

$$\eta_2 = \beta_{2,1}\,\eta_1 + \gamma_{2,1}\,\xi_1 + \gamma_{2,2}\,\xi_2 + \zeta_2$$

Ex. 4.1: Structural Equation Model with LISREL

Measurement Models for the *endogenous* latent variables:

$$\left.\begin{aligned} Y_1 - \mu_{Y_1} &= \lambda^Y_{1,1}\,\eta_1 + \varepsilon_1 \\ Y_2 - \mu_{Y_2} &= \lambda^Y_{2,1}\,\eta_1 + \varepsilon_2 \end{aligned}\right\} \text{ measurement model for } \eta_1$$

$$\left.\begin{aligned} Y_3 - \mu_{Y_3} &= \lambda^Y_{3,2}\,\eta_2 + \varepsilon_3 \\ Y_4 - \mu_{Y_4} &= \lambda^Y_{4,2}\,\eta_2 + \varepsilon_4 \end{aligned}\right\} \text{ measurement model for } \eta_2$$

Measurement Models for the *exogenous* latent variables:

$$X_1 - \mu_{X_1} = \lambda^X_{1,1}\,\xi_1 + \delta_1 \qquad \text{(measurement model for } \xi_1\text{)}$$

$$X_2 - \mu_{X_2} = \lambda^X_{2,2}\,\xi_2 + \delta_2 \qquad \text{(measurement model for } \xi_2\text{)}$$

We note that here we have only *reflective measurement* models.

Unlike in this example, the exogenous latent variables could (and usually will) also have *more than one* measurement variable.

Ex. 4.1: Structural Equation Model with LISREL

Finally we write the models in *matrix notation*.

From Ex. 3.1 we find for the *structural (inner) model*:

$$\boldsymbol{\eta} = \mathbf{B}\,\boldsymbol{\eta} + \boldsymbol{\Gamma}\,\boldsymbol{\xi} + \boldsymbol{\zeta}$$

with

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \qquad \boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \qquad \boldsymbol{\zeta} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix}, \qquad \boldsymbol{\Gamma} = \begin{pmatrix} \gamma_{1,1} & 0 \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix}.$$

Explicitly, we have the matrix equation

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 & \beta_{1,2} \\ \beta_{2,1} & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \gamma_{1,1} & 0 \\ \gamma_{2,1} & \gamma_{2,2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}.$$

Ex. 4.1: Structural Equation Model with LISREL

For the *exogenous* latent variables, the *measurement model in matrix notation* reads:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \begin{pmatrix} \mu_{X_1} \\ \mu_{X_2} \end{pmatrix} = \begin{pmatrix} \lambda^X_{1,1} & 0 \\ 0 & \lambda^X_{2,2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$$

or in shorter notation

$$\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}} = \boldsymbol{\Lambda}_X \, \boldsymbol{\xi} + \boldsymbol{\delta}$$

with

$$\mathbf{x} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \qquad \boldsymbol{\mu}_{\mathbf{x}} = \begin{pmatrix} \mu_{X_1} \\ \mu_{X_2} \end{pmatrix}, \qquad \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

$$\boldsymbol{\delta} = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}, \qquad \boldsymbol{\Lambda}_X = \begin{pmatrix} \lambda^X_{1,1} & 0 \\ 0 & \lambda^X_{2,2} \end{pmatrix}.$$

Ex. 4.1: Structural Equation Model with LISREL

For the *endogenous* latent variables, the *measurement model in matrix notation* reads:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} - \begin{pmatrix} \mu_{Y_1} \\ \mu_{Y_2} \\ \mu_{Y_3} \\ \mu_{Y_4} \end{pmatrix} = \begin{pmatrix} \lambda^Y_{1,1} & 0 \\ \lambda^Y_{2,1} & 0 \\ 0 & \lambda^Y_{3,2} \\ 0 & \lambda^Y_{4,2} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$

or in shorter notation

$$\mathbf{y} - \boldsymbol{\mu}_{\mathbf{y}} = \boldsymbol{\Lambda}_Y \, \boldsymbol{\eta} + \boldsymbol{\varepsilon}$$

with $\boldsymbol{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ and

$$\mathbf{y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}, \quad \boldsymbol{\mu}_{\mathbf{y}} = \begin{pmatrix} \mu_{Y_1} \\ \mu_{Y_2} \\ \mu_{Y_3} \\ \mu_{Y_4} \end{pmatrix}, \quad \boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}, \quad \boldsymbol{\Lambda}_Y = \begin{pmatrix} \lambda^Y_{1,1} & 0 \\ \lambda^Y_{2,1} & 0 \\ 0 & \lambda^Y_{3,2} \\ 0 & \lambda^Y_{4,2} \end{pmatrix}.$$

Ex. 4.2: Empirical Covariance Matrix

Given the data below for the measurement variables X_1 = *yearly salary in 1000 Euros*, Y_1 = *average hours of work per week*, Y_2 = *average number of lines of code per week (measured in units of 100 lines of code)*, for a software programmer on a non-pay-scale salary, compute the *empirical covariance matrix* **S**.

Programmer	X_1	Y_1	Y_2
e_1	50	45	50
e_2	60	55	55
e_3	70	50	60

Solution: We start by computing the *means* of the data of the measurement variables:

$$\overline{x_1} = \frac{1}{3}\left(50 + 60 + 70\right) = \frac{180}{3} = 60,$$

Ex. 4.2: Empirical Covariance Matrix

$$\overline{y_1} = \frac{1}{3}\left(45+55+50\right) = \frac{150}{3} = 50,$$

$$\overline{y_2} = \frac{1}{3}\left(50+55+60\right) = \frac{165}{3} = 55.$$

Hence the *expectation values* μ_{X_1}, μ_{Y_1} and μ_{Y_2} *are estimated by* $\overline{x_1} = 60$, $\overline{y_1} = 50$ and $\overline{y_2} = 55$. Now we can write down the *centered data matrix*

$$\mathbf{W} = \begin{pmatrix} x_{1,1} - \overline{x_1} & y_{1,1} - \overline{y_1} & y_{1,2} - \overline{y_2} \\ x_{2,1} - \overline{x_1} & y_{2,1} - \overline{y_1} & y_{2,2} - \overline{y_2} \\ x_{3,1} - \overline{x_1} & y_{3,1} - \overline{y_1} & y_{3,2} - \overline{y_2} \end{pmatrix}$$

$$= \begin{pmatrix} 50-60 & 45-50 & 50-55 \\ 60-60 & 55-50 & 55-55 \\ 70-60 & 50-50 & 60-55 \end{pmatrix} = \begin{pmatrix} -10 & -5 & -5 \\ 0 & 5 & 0 \\ 10 & 0 & 5 \end{pmatrix}.$$

Ex. 4.2: Empirical Covariance Matrix

The *empirical covariance matrix* is given by:

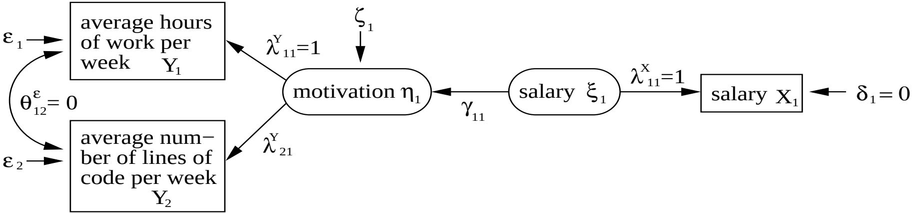
$$\mathbf{S} = \frac{1}{3-1}\mathbf{W}'\mathbf{W} = \frac{1}{2}\begin{pmatrix} -10 & 0 & 10 \\ -5 & 5 & 0 \\ -5 & 0 & 5 \end{pmatrix}\begin{pmatrix} -10 & -5 & -5 \\ 0 & 5 & 0 \\ 10 & 0 & 5 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 100+0+100 & 50+0+0 & 50+0+50 \\ 50+0+0 & 25+25+0 & 25+0+0 \\ 50+0+50 & 25+0+0 & 25+0+25 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 200 & 50 & 100 \\ 50 & 50 & 25 \\ 100 & 25 & 50 \end{pmatrix} = \begin{pmatrix} 100 & 25 & 50 \\ 25 & 25 & 12.5 \\ 50 & 12.5 & 25 \end{pmatrix}$$

Ex. 4.3: Solving a Structural Equation Model with LISREL

To demonstrate the solution of a structural equation model with *LISREL*, we consider the simplified model for the *work of a software programmer on a non-pay-scale salary* shown in the diagram below.



1. Set up the structural equation model by specifying the *structural (inner) model* and the *measurement model*.
2. *Determine* with the LISREL approach *the model parameters* in terms of the covariances of the measurement variables.
3. Use the empirical covariance matrix from Ex. 4.2 to compute the *numerical values for the parameters* and interpret your results.

Ex. 4.3: Solving a Structural Equation Model with LISREL

<u>Solution:</u> The *structural (inner) model* consists here of the one equation:

$$\eta_1 = \gamma_{1,1}\,\xi_1 + \zeta_1 \tag{8}$$

The *measurement model* consists of the three equations:

$$X_1 - \mu_{X_1} = \xi_1 \qquad \text{(since } \delta_1 = 0 \text{ and } \lambda^X_{1,1} = 1\text{)} \tag{9}$$

$$Y_1 - \mu_{Y_1} = \eta_1 + \varepsilon_1 \qquad \text{(sine } \lambda^Y_{1,1} = 1\text{)} \tag{10}$$

$$Y_2 - \mu_{Y_2} = \lambda^Y_{2,1}\,\eta_1 + \varepsilon_2 \tag{11}$$

We note that $\delta_1 = 0$ and $\lambda^X_{1,1} = 1$ are chosen because the latent variable $\xi_1 =$ salary is measured directly and without error (hence $\delta_1 = 0$). Hence ξ_1 automatically has a scale. The choice $\lambda^Y_{1,1} = 1$ however, is simply made to give the latent variable η_1 a scale.

Apart from these equations we are given the information that *the error terms ε_1 and ε_2 of Y_1 and Y_2, respectively, are uncorrelated*, since

$$\theta^\varepsilon_{1,2} = \mathrm{Cov}(\varepsilon_1, \varepsilon_2) = 0. \tag{12}$$

Ex. 4.3: Solving a Structural Equation Model with LISREL

We have $q = 1$ measurement variables X_1 for ξ_1, and we have $p = 2$ measurement variables Y_1 and Y_2 for η_1. Hence we get

$$\frac{(p+q)(p+q+1)}{2} = \frac{(2+1)(2+1+1)}{2} = \frac{12}{2} = 6$$

different entries in the covariance matrix of the measurement variables. These *6 different entries in the covariance matrix* are the *variances*

$$\text{Var}(X_1), \qquad \text{Var}(Y_1) \qquad \text{and} \qquad \text{Var}(Y_2), \tag{13}$$

and the *covariances*

$$\text{Cov}(X_1, Y_1), \qquad \text{Cov}(X_1, Y_2) \qquad \text{and} \qquad \text{Cov}(Y_1, Y_2). \tag{14}$$

Inspecting our model (see the diagram) we find that we have also *6 unknown model parameters*: From the structural (inner) model we have the parameters

$$\gamma_{1,1}, \qquad \phi_{1,1} = \text{Var}(\xi_1) \qquad \text{and} \qquad \psi_{1,1} = \text{Var}(\zeta_1)$$

and from the (outer) measurement model we have the parameters

$$\lambda^X_{2,1}, \qquad \theta^\varepsilon_{1,1} = \mathrm{Var}(\varepsilon_1) \qquad \text{and} \qquad \theta^\varepsilon_{2,2} = \mathrm{Var}(\varepsilon_2).$$

Normally we would also have to consider the parameter $\theta^\delta_{1,1} = \mathrm{Var}(\delta_1)$, but since $\delta_1 = 0$ (as ξ_1 is measured exactly) we clearly have

$$\theta^\delta_{1,1} = \mathrm{Var}(\delta_1) = 0. \tag{15}$$

As we have 6 unknown model parameters and also 6 different entries in the covariance matrix, *our LISREL model could be identifiable*.

Next we use the equations (8) to (11), as well as the additional information from (12) and (15), to *compute the entries (13) and (14) of the covariance matrix in terms of the model parameters*.

Afterwards we will try to *solve these 6 equations for the model parameters*.

Ex. 4.3: Solving a Structural Equation Model with LISREL

Before we start to compute the variances (13) and the covariances (14), we remember the assumptions from the LISREL model for our concrete example:

$$\mathsf{E}(\xi_1) = 0, \qquad \mathsf{E}(\eta_1) = 0, \qquad \mathsf{Cov}(\xi_1, \zeta_1) = 0, \tag{16}$$

$$\mathsf{E}(\zeta_1) = 0, \qquad \mathsf{E}(\varepsilon_1) = 0, \qquad \mathsf{E}(\varepsilon_2) = 0, \tag{17}$$

$$\mathsf{Cov}(\varepsilon_1, \eta_1) = 0, \qquad \mathsf{Cov}(\varepsilon_2, \eta_1) = 0, \qquad \mathsf{Cov}(\varepsilon_1, \xi_1) = 0, \tag{18}$$

$$\mathsf{Cov}(\varepsilon_2, \xi_1) = 0, \qquad \mathsf{Cov}(\varepsilon_1, \zeta_1) = 0, \qquad \mathsf{Cov}(\varepsilon_2, \zeta_1) = 0, \tag{19}$$

From (9) we have

$$\mathsf{Var}(X_1) = \mathsf{E}\big([X_1 - \mu_{X_1}]^2\big) = \mathsf{E}(\xi_1^2) = \mathsf{E}\big([\xi_1 - \underbrace{\mathsf{E}(\xi_1)}_{=0}]^2\big) = \mathsf{Var}(\xi_1) = \phi_{1,1},$$

where we have used the first equation in (16) in the second last step. This identifies the model parameter $\phi_{1,1} = \mathsf{Var}(\xi_1)$ uniquely:

$$\phi_{1,1} = \mathsf{Var}(\xi_1) = \mathsf{Var}(X_1). \tag{20}$$

Ex. 4.3: Solving a Structural Equation Model with LISREL

From (10) we have

$$
\begin{aligned}
\mathrm{Var}(Y_1) &= \mathrm{E}\big([Y_1 - \mu_{Y_1}]^2\big) = \mathrm{E}\big([\eta_1 + \varepsilon_1]^2\big) = \mathrm{E}\big(\eta_1^2 + 2\,\eta_1\,\varepsilon_1 + \varepsilon_1^2\big) \\
&= \mathrm{E}(\eta_1^2) + 2\,\mathrm{E}(\eta_1\,\varepsilon_1) + \mathrm{E}(\varepsilon_1^2) \\
&= \mathrm{E}\big([\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]^2\big) + 2\,\mathrm{E}\big([\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]\,[\varepsilon_1 - \underbrace{\mathrm{E}(\varepsilon_1)}_{=0}]\big) + \mathrm{E}\big([\varepsilon_1 - \underbrace{\mathrm{E}(\varepsilon_1)}_{=0}]^2\big) \\
&= \mathrm{Var}(\eta_1) + 2\,\underbrace{\mathrm{Cov}(\eta_1, \varepsilon_1)}_{=0} + \mathrm{Var}(\varepsilon_1) \\
&= \mathrm{Var}(\eta_1) + \mathrm{Var}(\varepsilon_1) = \mathrm{Var}(\eta_1) + \theta^{\varepsilon}_{1,1}
\end{aligned}
$$

where we have used the linearity of the expectation value and the assumptions (16) to (18). So we have found the equation

$$\mathrm{Var}(Y_1) = \mathrm{Var}(\eta_1) + \theta^{\varepsilon}_{1,1} \tag{21}$$

which contains an additional unknown $\mathrm{Var}(\eta_1)$ that we need to eliminate when we determine our parameters.

Ex. 4.3: Solving a Structural Equation Model with LISREL

From (11) we have

$$
\begin{aligned}
\mathrm{Var}(Y_2) &= \mathrm{E}\big([Y_2 - \mu_{Y_2}]^2\big) = \mathrm{E}\big([\lambda^Y_{2,1}\,\eta_1 + \varepsilon_2]^2\big) \\
&= \mathrm{E}\big((\lambda^Y_{2,1})^2\,\eta_1^2 + 2\,\lambda^Y_{2,1}\,\eta_1\,\varepsilon_2 + \varepsilon_2^2\big) \\
&= (\lambda^Y_{2,1})^2\,\mathrm{E}(\eta_1^2) + 2\,\lambda^Y_{2,1}\,\mathrm{E}(\eta_1\,\varepsilon_2) + \mathrm{E}(\varepsilon_2^2) \\
&= (\lambda^Y_{2,1})^2\,\mathrm{E}\big([\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]^2\big) + 2\,\lambda^Y_{2,1}\,\mathrm{E}\big([\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]\,[\varepsilon_2 - \underbrace{\mathrm{E}(\varepsilon_2)}_{=0}]\big) \\
&\quad + \mathrm{E}\big([\varepsilon_2 - \underbrace{\mathrm{E}(\varepsilon_2)}_{=0}]^2\big) \\
&= (\lambda^Y_{2,1})^2\,\mathrm{Var}(\eta_1) + 2\,\lambda^Y_{2,1}\,\underbrace{\mathrm{Cov}(\eta_1,\varepsilon_2)}_{=0} + \mathrm{Var}(\varepsilon_2) \\
&= (\lambda^Y_{2,1})^2\,\mathrm{Var}(\eta_1) + \mathrm{Var}(\varepsilon_2) = (\lambda^Y_{2,1})^2\,\mathrm{Var}(\eta_1) + \theta^\varepsilon_{2,2},
\end{aligned}
$$

where we have used the linearity of the expectation value and the assumptions (16) to (18).

Ex. 4.3: Solving a Structural Equation Model with LISREL

So we have found the equation

$$\mathrm{Var}(Y_2) = (\lambda^Y_{2,1})^2\,\mathrm{Var}(\eta_1) + \theta^\varepsilon_{2,2} \tag{22}$$

which also contains the additional unknown $\mathrm{Var}(\eta_1)$ that we need to eliminate when we determine our parameters.

From (9) and (10) we have

$$\begin{aligned}
&\mathrm{Cov}(X_1, Y_1) = \mathrm{E}\big([X_1 - \mu_{X_1}]\,[Y_1 - \mu_{Y_1}]\big) = \mathrm{E}\big(\xi_1\,[\eta_1 + \varepsilon_1]\big) \\
&= \mathrm{E}\big(\xi_1\,\eta_1 + \xi_1\,\varepsilon_1\big) = \mathrm{E}(\xi_1\,\eta_1) + \mathrm{E}(\xi_1\,\varepsilon_1) \\
&= \mathrm{E}\big([\xi_1 - \underbrace{\mathrm{E}(\xi_1)}_{=0}]\,[\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]\big) + \mathrm{E}([\xi_1 - \underbrace{\mathrm{E}(\xi_1)}_{=0}]\,[\varepsilon_1 - \underbrace{\mathrm{E}(\varepsilon_1)}_{=0}]) \\
&= \mathrm{Cov}(\xi_1, \eta_1) + \underbrace{\mathrm{Cov}(\xi_1, \varepsilon_1)}_{=0} = \mathrm{Cov}(\xi_1, \eta_1),
\end{aligned}$$

where we have used the linearity of the expectation value and the assumptions (16) to (18).

Ex. 4.3: Solving a Structural Equation Model with LISREL

So we have found the equation

$$\mathrm{Cov}(X_1, Y_1) = \mathrm{Cov}(\xi_1, \eta_1), \tag{23}$$

with the unknown $\mathrm{Cov}(\xi_1, \eta_1)$ that we still need to eliminate when we determine the model parameters.

From (9) and (11) we find

$$\begin{aligned}
\mathrm{Cov}(X_1, Y_2) &= \mathrm{E}\big([X_1 - \mu_{X_1}]\,[Y_1 - \mu_{Y_2}]\big) = \mathrm{E}\big(\xi_1\,[\lambda^Y_{2,1}\,\eta_1 + \varepsilon_2]\big) \\
&= \mathrm{E}\big(\lambda^Y_{2,1}\,\xi_1\,\eta_1 + \xi_1\,\varepsilon_2\big) = \lambda^Y_{2,1}\,\mathrm{E}(\xi_1\,\eta_1) + \mathrm{E}(\xi_1\,\varepsilon_2) \\
&= \lambda^Y_{2,1}\,\mathrm{E}\big([\xi_1 - \underbrace{\mathrm{E}(\xi_1)}_{=0}]\,[\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]\big) + \mathrm{E}\big([\xi_1 - \underbrace{\mathrm{E}(\xi_1)}_{=0}]\,[\varepsilon_2 - \underbrace{\mathrm{E}(\varepsilon_2)}_{=0}]\big) \\
&= \lambda^Y_{2,1}\,\mathrm{Cov}(\xi_1, \eta_1) + \underbrace{\mathrm{Cov}(\xi_1, \varepsilon_2)}_{=0} = \lambda^Y_{2,1}\,\mathrm{Cov}(\xi_1, \eta_1),
\end{aligned}$$

where we have used the linearity of the expectation value and the assumptions (16), (17) and (19).

Ex. 4.3: Solving a Structural Equation Model with LISREL

So we have found the equation

$$\mathrm{Cov}(X_1, Y_2) = \lambda_{2,1}^{Y} \,\mathrm{Cov}(\xi_1, \eta_1), \tag{24}$$

with the unknown $\mathrm{Cov}(\xi_1, \eta_1)$ that we still need to eliminate when we determine the model parameters.

From (10) and (11) we find

$$\begin{aligned}
\mathrm{Cov}(Y_1, Y_2) &= \mathrm{E}\big([Y_1 - \mu_{Y_1}]\,[Y_2 - \mu_{Y_2}]\big) = \mathrm{E}\big([\eta_1 + \varepsilon_1]\,[\lambda_{2,1}^{Y}\,\eta_1 + \varepsilon_2]\big) \\
&= \mathrm{E}\big(\lambda_{2,1}^{Y}\,\eta_1^2 + \eta_1\,\varepsilon_2 + \lambda_{2,1}^{Y}\,\varepsilon_1\,\eta_1 + \varepsilon_1\,\varepsilon_2\big) \\
&= \lambda_{2,1}^{Y}\,\mathrm{E}(\eta_1^2) + \mathrm{E}(\eta_1\,\varepsilon_2) + \lambda_{2,1}^{Y}\,\mathrm{E}(\varepsilon_1\,\eta_1) + \mathrm{E}(\varepsilon_1\,\varepsilon_2) \\
&= \lambda_{2,1}^{Y}\,\mathrm{E}\big([\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]^2\big) + \mathrm{E}([\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]\,[\varepsilon_2 - \underbrace{\mathrm{E}(\varepsilon_2)}_{=0}]) \\
&\quad + \lambda_{2,1}^{Y}\,\mathrm{E}\big([\varepsilon_1 - \underbrace{\mathrm{E}(\varepsilon_1)}_{=0}]\,[\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]\big) + \mathrm{E}([\varepsilon_1 - \underbrace{\mathrm{E}(\varepsilon_1)}_{=0}]\,[\varepsilon_2 - \underbrace{\mathrm{E}(\varepsilon_2)}_{=0}])
\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}(Y_1, Y_2) &= \lambda^Y_{2,1}\,\mathrm{Var}(\eta_1) + \underbrace{\mathrm{Cov}(\eta_1, \varepsilon_2)}_{=0} + \lambda^Y_{2,1}\,\underbrace{\mathrm{Cov}(\varepsilon_1, \eta_1)}_{=0} + \underbrace{\mathrm{Cov}(\varepsilon_1, \varepsilon_2)}_{=\theta^\varepsilon_{1,2}=0} \\ &= \lambda^Y_{2,1}\,\mathrm{Var}(\eta_1),\end{aligned}$$

where we have used the linearity of the expectation value, the condition (12) and the assumptions (16) to (18).
So we have found the equation

$$\mathrm{Cov}(Y_1, Y_2) = \lambda^Y_{2,1}\,\mathrm{Var}(\eta_1), \tag{25}$$

with the unknown $\mathrm{Var}(\eta_1)$ that we still need to eliminate when we determine the model parameters.

We summarize the *6 equations for the covariance of the measurement variables* on the next slide.

Then *use equation (8) from the structural (inner) model to first compute and then eliminate the unknowns* $\mathrm{Var}(\eta_1)$ and $\mathrm{Cov}(\xi_1, \eta_1)$.

From (20) to (25) we have the 6 equations:

$$\begin{aligned}
\mathrm{Var}(X_1) &= \phi_{1,1} && (26)\\
\mathrm{Var}(Y_1) &= \mathrm{Var}(\eta_1) + \theta^{\varepsilon}_{1,1} && (27)\\
\mathrm{Var}(Y_2) &= (\lambda^{Y}_{2,1})^2\,\mathrm{Var}(\eta_1) + \theta^{\varepsilon}_{2,2} && (28)\\
\mathrm{Cov}(X_1, Y_1) &= \mathrm{Cov}(\xi_1, \eta_1) && (29)\\
\mathrm{Cov}(X_1, Y_2) &= \lambda^{Y}_{2,1}\,\mathrm{Cov}(\xi_1, \eta_1) && (30)\\
\mathrm{Cov}(Y_1, Y_2) &= \lambda^{Y}_{2,1}\,\mathrm{Var}(\eta_1) && (31)
\end{aligned}$$

Next we use the equation (8) from the structural (inner) model to compute $\mathrm{Var}(\eta_1)$ and $\mathrm{Cov}(\xi_1, \eta_1)$:

$$
\begin{aligned}
\mathrm{Var}(\eta_1) &= \mathsf{E}\big([\eta_1 - \underbrace{\mathsf{E}(\eta_1)}_{=0}]^2\big) = \mathsf{E}(\eta_1^2) = \mathsf{E}\big([\gamma_{1,1}\,\xi_1 + \zeta_1]^2\big) \\
&= \mathsf{E}\big(\gamma_{1,1}^2\,\xi_1^2 + 2\,\gamma_{1,1}\,\xi_1\,\zeta_1 + \zeta_1^2\big) \\
&= \gamma_{1,1}^2\,\mathsf{E}(\xi_1^2) + 2\,\gamma_{1,1}\,\mathsf{E}(\xi_1\,\zeta_1) + \mathsf{E}(\zeta_1^2) \\
&= \gamma_{1,1}^2\,\mathsf{E}\big([\xi_1 - \underbrace{\mathsf{E}(\xi_1)}_{=0}]^2\big) + 2\,\gamma_{1,1}\,\mathsf{E}\big([\xi_1 - \underbrace{\mathsf{E}(\xi_1)}_{=0}]\,[\zeta_1 - \underbrace{\mathsf{E}(\zeta_1)}_{=0}]\big) \\
&\quad + \mathsf{E}\big([\zeta - \underbrace{\mathsf{E}(\zeta_1)}_{=0}]^2\big) \\
&= \gamma_{1,1}^2\,\underbrace{\mathrm{Var}(\xi_1)}_{=\phi_{1,1}} + 2\,\gamma_{1,1}\,\underbrace{\mathrm{Cov}(\xi_1,\zeta_1)}_{=0} + \underbrace{\mathrm{Var}(\zeta_1)}_{=\psi_{1,1}} \\
&= \gamma_{1,1}^2\,\phi_{1,1} + \psi_{1,1},
\end{aligned}
$$

where we have used the linearity of the expectation value and the assumptions (16) and (17).

Ex. 4.3: Solving a Structural Equation Model with LISREL

$$\begin{aligned}\mathrm{Cov}(\xi_1,\eta_1) &= \mathrm{E}\big([\xi_1 - \underbrace{\mathrm{E}(\xi_1)}_{=0}]\,[\eta_1 - \underbrace{\mathrm{E}(\eta_1)}_{=0}]\big) = \mathrm{E}(\xi_1\,\eta_1)\\ &= \mathrm{E}\big(\xi_1\,[\gamma_{1,1}\,\xi_1 + \zeta_1]\big) = \mathrm{E}\big(\gamma_{1,1}\,\xi_1^2 + \xi_1\,\zeta_1\big) = \gamma_{1,1}\,\mathrm{E}(\xi_1^2) + \mathrm{E}(\xi_1\,\zeta_1)\\ &= \gamma_{1,1}\,\mathrm{E}\big([\xi_1 - \underbrace{\mathrm{E}(\xi_1)}_{=0}]^2\big) + \mathrm{E}\big([\xi_1 - \underbrace{\mathrm{E}(\xi_1)}_{=0}]\,[\zeta_1 - \underbrace{\mathrm{E}(\zeta_1)}_{=0}]\big)\\ &= \gamma_{1,1}\,\underbrace{\mathrm{Var}(\xi_1)}_{=\phi_{1,1}} + \underbrace{\mathrm{Cov}(\xi_1,\zeta_1)}_{=0} = \gamma_{1,1}\,\phi_{1,1},\end{aligned}$$

where we have used the linearity of the expectation value and the assumptions (16) and (17).

So in addition to (26) to (31), we have found:

$$\mathrm{Var}(\eta_1) = \gamma_{1,1}^2\,\phi_{1,1} + \psi_{1,1} \qquad (32)$$

$$\mathrm{Cov}(\xi_1,\eta_1) = \gamma_{1,1}\,\phi_{1,1} \qquad (33)$$

Ex. 4.3: Solving a Structural Equation Model with LISREL

We have *8 equations (26) to (33) and 8 unknown parameters* $\gamma_{1,1}$, $\phi_{1,1}$, $\psi_{1,1}$, $\lambda^Y_{2,1}$, $\theta^\varepsilon_{1,1}$, $\theta^\varepsilon_{2,2}$ and $\mathrm{Var}(\eta_1)$, $\mathrm{Cov}(\xi_1, \eta_1)$. First we note that from (26)

$$\phi_{1,1} = \mathrm{Var}(X_1). \tag{34}$$

Then we substitute the expression for $\mathrm{Cov}(\xi_1, \eta_1)$ from (33) into (29) and subsequently use (34)

$$\mathrm{Cov}(X_1, Y_1) = \gamma_{1,1}\,\phi_{1,1} = \gamma_{1,1}\,\mathrm{Var}(X_1).$$

Hence, we get

$$\gamma_{1,1} = \frac{\mathrm{Cov}(X_1, Y_1)}{\mathrm{Var}(X_1)}. \tag{35}$$

Then we substitute in (30) $\mathrm{Cov}(\xi_1, \eta_1)$ by (29) and get

$$\mathrm{Cov}(X_1, Y_2) = \lambda^Y_{2,1}\,\mathrm{Cov}(X_1, Y_1).$$

Ex. 4.3: Solving a Structural Equation Model with LISREL

Hence, we get

$$\lambda^Y_{2,1} = \frac{\mathrm{Cov}(X_1, Y_2)}{\mathrm{Cov}(X_1, Y_1)}. \tag{36}$$

Next we substitute $\mathrm{Var}(\eta_1)$ in (31) by (32)

$$\mathrm{Cov}(Y_1, Y_2) = \lambda^Y_{2,1}\left[\gamma^2_{1,1}\,\phi_{1,1} + \psi_{1,1}\right] = \lambda^Y_{2,1}\,\gamma^2_{1,1}\,\phi_{1,1} + \lambda^Y_{2,1}\,\psi_{1,1}$$

and solve for $\psi_{1,1}$

$$\psi_{1,1} = \frac{1}{\lambda^Y_{2,1}}\left[\mathrm{Cov}(Y_1, Y_2) - \lambda^Y_{2,1}\,\gamma^2_{1,1}\,\phi_{1,1}\right] = \frac{\mathrm{Cov}(Y_1, Y_2)}{\lambda^Y_{2,1}} - \gamma^2_{1,1}\,\phi_{1,1}. \tag{37}$$

Now we use (34), (35) and (36) to eliminate all the other parameters in (37):

$$\psi_{1,1} = \frac{\mathrm{Cov}(X_1, Y_1)}{\mathrm{Cov}(X_1, Y_2)}\,\mathrm{Cov}(Y_1, Y_2) - \left(\frac{\mathrm{Cov}(X_1, Y_1)}{\mathrm{Var}(X_1)}\right)^2 \mathrm{Var}(X_1),$$

Ex. 4.3: Solving a Structural Equation Model with LISREL

and simplifying we find

$$\psi_{1,1} = \frac{\mathrm{Cov}(X_1, Y_1)}{\mathrm{Cov}(X_1, Y_2)} \mathrm{Cov}(Y_1, Y_2) - \frac{[\mathrm{Cov}(X_1, Y_1)]^2}{\mathrm{Var}(X_1)}. \qquad (38)$$

Next we solve (27) for $\theta^{\varepsilon}_{1,1}$ and subsequently substitute $\mathrm{Var}(\eta)$ by (32)

$$\theta^{\varepsilon}_{1,1} = \mathrm{Var}(Y_1) - \mathrm{Var}(\eta_1). \qquad (39)$$

We note that from rearranging (31)

$$\mathrm{Var}(\eta_1) = \frac{\mathrm{Cov}(Y_1, Y_2)}{\lambda^Y_{2,1}}. \qquad (40)$$

Substituting (40) and subsequently (36) into (39) yields

$$\theta^{\varepsilon}_{1,1} = \mathrm{Var}(Y_1) - \frac{\mathrm{Cov}(Y_1, Y_2)}{\lambda^Y_{2,1}} = \mathrm{Var}(Y_1) - \frac{\mathrm{Cov}(Y_1, Y_2)\,\mathrm{Cov}(X_1, Y_1)}{\mathrm{Cov}(X_1, Y_2)},$$

and hence

$$\theta^{\varepsilon}_{1,1} = \mathrm{Var}(Y_1) - \frac{\mathrm{Cov}(Y_1, Y_2)\,\mathrm{Cov}(X_1, Y_1)}{\mathrm{Cov}(X_1, Y_2)}. \tag{41}$$

Finally from solving (28) for $\theta^{\varepsilon}_{2,2}$ we get

$$\theta^{\varepsilon}_{2,2} = \mathrm{Var}(Y_2) - (\lambda^{Y}_{2,1})^2\,\mathrm{Var}(\eta_1). \tag{42}$$

Substituting (40) into (42) yields

$$\theta^{\varepsilon}_{2,2} = \mathrm{Var}(Y_2) - (\lambda^{Y}_{2,1})^2\,\frac{\mathrm{Cov}(Y_1, Y_2)}{\lambda^{Y}_{2,1}} = \mathrm{Var}(Y_2) - \lambda^{Y}_{2,1}\,\mathrm{Cov}(Y_1, Y_2),$$

and finally substituting $\lambda^{Y}_{2,1}$ buy (36) yields

$$\theta^{\varepsilon}_{2,2} = \mathrm{Var}(Y_2) - \frac{\mathrm{Cov}(X_1, Y_2)\,\mathrm{Cov}(Y_1, Y_2)}{\mathrm{Cov}(X_1, Y_1)}. \tag{43}$$

Ex. 4.3: Solving a Structural Equation Model with LISREL

We *summarize the formulas* (34), (35), (36), (38), (41) and (43) that *identify the 6 model parameters*:

$$\phi_{1,1} = \mathrm{Var}(X_1) \tag{44}$$

$$\gamma_{1,1} = \frac{\mathrm{Cov}(X_1, Y_1)}{\mathrm{Var}(X_1)} \tag{45}$$

$$\lambda^{Y}_{2,1} = \frac{\mathrm{Cov}(X_1, Y_2)}{\mathrm{Cov}(X_1, Y_1)} \tag{46}$$

$$\psi_{1,1} = \frac{\mathrm{Cov}(X_1, Y_1)}{\mathrm{Cov}(X_1, Y_2)}\,\mathrm{Cov}(Y_1, Y_2) - \frac{[\mathrm{Cov}(X_1, Y_1)]^2}{\mathrm{Var}(X_1)} \tag{47}$$

$$\theta^{\varepsilon}_{1,1} = \mathrm{Var}(Y_1) - \frac{\mathrm{Cov}(Y_1, Y_2)\,\mathrm{Cov}(X_1, Y_1)}{\mathrm{Cov}(X_1, Y_2)} \tag{48}$$

$$\theta^{\varepsilon}_{2,2} = \mathrm{Var}(Y_2) - \frac{\mathrm{Cov}(X_1, Y_2)\,\mathrm{Cov}(Y_1, Y_2)}{\mathrm{Cov}(X_1, Y_1)} \tag{49}$$

Ex. 4.3: Solving a Structural Equation Model with LISREL

Finally the *empirical covariance matrix from Ex. 4.2* is given by

$$\mathbf{S} = \begin{pmatrix} \widehat{\mathrm{Var}}(X_1) & \widehat{\mathrm{Cov}}(X_1, Y_1) & \widehat{\mathrm{Cov}}(X_1, Y_2) \\ \widehat{\mathrm{Cov}}(Y_1, X_1) & \widehat{\mathrm{Var}}(Y_1) & \widehat{\mathrm{Cov}}(Y_1, Y_2) \\ \widehat{\mathrm{Cov}}(Y_2, X_1) & \widehat{\mathrm{Cov}}(Y_2, Y_1) & \widehat{\mathrm{Var}}(Y_2) \end{pmatrix}$$

$$= \begin{pmatrix} 100 & 25 & 50 \\ 25 & 25 & 12.5 \\ 50 & 12.5 & 25 \end{pmatrix}.$$

Thus the variances and covariances are estimated by:

$$\widehat{\mathrm{Var}}(X_1) = 100, \quad \widehat{\mathrm{Var}}(Y_1) = 25, \quad \widehat{\mathrm{Var}}(Y_2) = 25, \tag{50}$$

$$\widehat{\mathrm{Cov}}(X_1, Y_1) = 25, \quad \widehat{\mathrm{Cov}}(X_1, Y_2) = 50, \quad \widehat{\mathrm{Cov}}(Y_1, Y_2) = 12.5. \tag{51}$$

Substituting the estimated values of the variances and covariances in (50) and (51) into (44) to (49) yields:

Ex. 4.3: Solving a Structural Equation Model with LISREL

$$\phi_{1,1} = 100$$

$$\gamma_{1,1} = \frac{25}{100} = \frac{1}{4} = 0.25$$

$$\lambda^{Y}_{2,1} = \frac{50}{12.5} = 2$$

$$\psi_{1,1} = \frac{25 \cdot 12.5}{50} - \frac{(25)^2}{100} = 6.25 - \frac{25}{4} = 6.25 - 6.25 = 0$$

$$\theta^{\varepsilon}_{1,1} = 25 - \frac{12.5 \cdot 25}{50} = 25 - 6.25 = 18.75$$

$$\theta^{\varepsilon}_{2,2} = 25 - \frac{50 \cdot 12.5}{25} = 25 - 25 = 0$$

Inspecting the model parameters briefly, we note that we have no negative variances, since $\phi_{1,1} = \text{Var}(\xi_1) = 100$, $\psi_{1,1} = \text{Var}(\zeta_1) = 0$, $\theta^{\varepsilon}_{1,1} = \text{Var}(\varepsilon_1) = 18.75$ and $\theta^{\varepsilon}_{2,2} = 0$. So this makes sense.

Ex. 4.3: Solving a Structural Equation Model with LISREL

Next we inspect the path coefficients $\gamma_{1,1} = 0.25$ and $\lambda^{Y}_{2,1} = 2$ which are both positive. This makes sense, as our logical considerations tell us:

- The higher the salary, the higher we expect the motivation of the software programmer to be. Hence, $\gamma_{1,1}$ should be positive.
- The higher the motivation of the software programmer, the more lines of code we expect him to write per week. Hence, $\lambda^{Y}_{2,1}$ should be positive.

So our LISREL model *result coincides with our logical considerations*.

Now the model would have to be tested with model quality criteria that are beyond the scope of this course. Also, its is clear that our sample size was much to small to give representative results.

Structural Equation Modeling

Solutions to Topic 5: PLS Path Modeling (Partial Least Squares Path Modeling)

Dr. Kerstin Hesse

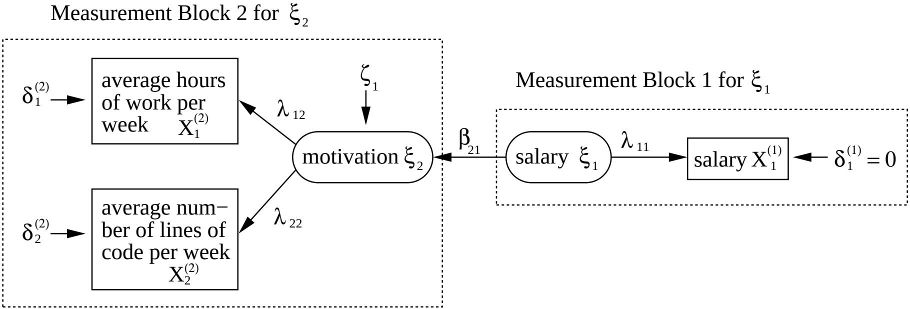
Email: kerstin.hesse@hhl.de; *Phone:* +49 (0)341 9851-820; *Office:* HHL Main Building, Room 115A

HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany

Doctoral Program at HHL, June 1-2, 2012

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

For the SEM in the example on pages 125–127 (which describes the work of a software programmer on a non-pay-scale salary) we were given the *SEM diagram* (in PLS notation) below



and we found the *model equations*:

$$\xi_2 = \beta_{2,1}\,\xi_1 + \zeta_1 \qquad \text{for the structural (inner) model,} \qquad (52)$$

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

and

$$X_1^{(1)} - \mu_{X_1^{(1)}} = \lambda_{1,1}\,\xi_1 + \delta_1^{(1)} \quad \text{for the measurement block for } \xi_1 \qquad (53)$$

$$\left.\begin{aligned} X_1^{(2)} - \mu_{X_1^{(2)}} &= \lambda_{1,2}\,\xi_2 + \delta_1^{(2)} \\ X_2^{(2)} - \mu_{X_2^{(2)}} &= \lambda_{2,2}\,\xi_2 + \delta_2^{(2)} \end{aligned}\right\} \text{for the measurement block for } \xi_2 \quad (54)$$

Now we are given the following data

Programmer	$X_1^{(1)}$	$X_1^{(2)}$	$X_2^{(2)}$
e_1	50	45	50
e_2	60	55	55
e_3	70	50	60

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

for the measurement variables: $X_1^{(1)}$ = yearly salary in 1000 Euros, $X_1^{(2)}$ = average hours of work per week, $X_2^{(2)}$ = average number of lines of code per week (measured in units of 100 lines of code). *Using equal weights as the initial weights, execute step 1 of the PLS algorithm.*

Solution: As a preparation we compute the *values of the centered data*:

Measurement Block 1 for ξ_1: We have the *mean*

$$\overline{x_1^{(1)}} = \frac{1}{3}\big(50 + 60 + 70\big) = \frac{180}{3} = 60,$$

and hence the *centered data for $X_1^{(1)}$* is given by

$$\begin{aligned} x_{1,1}^{(1)} - \overline{x_1^{(1)}} &= 50 - 60 = -10, \\ x_{2,1}^{(1)} - \overline{x_1^{(1)}} &= 60 - 60 = 0, \\ x_{3,1}^{(1)} - \overline{x_1^{(1)}} &= 70 - 60 = 10. \end{aligned} \tag{55}$$

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

Measurement Block 2 for ξ_2: We have the *means*

$$\overline{x_1^{(2)}} = \frac{1}{3}\left(45+55+50\right) = \frac{150}{3} = 50, \quad \overline{x_2^{(2)}} = \frac{1}{3}\left(50+55+60\right) = \frac{165}{3} = 55.$$

Hence the *centered data for* $X_1^{(2)}$ is given by

$$\begin{aligned} x_{1,1}^{(2)} - \overline{x_1^{(2)}} &= 45 - 50 = -5, \\ x_{2,1}^{(2)} - \overline{x_1^{(2)}} &= 55 - 50 = 5, \\ x_{3,1}^{(2)} - \overline{x_1^{(2)}} &= 50 - 50 = 0, \end{aligned} \tag{56}$$

and the *centered data for* $X_2^{(2)}$ is given by

$$\begin{aligned} x_{1,2}^{(2)} - \overline{x_2^{(2)}} &= 50 - 55 = -5, \\ x_{2,2}^{(2)} - \overline{x_2^{(2)}} &= 55 - 55 = 0, \\ x_{3,2}^{(2)} - \overline{x_2^{(2)}} &= 60 - 55 = 5. \end{aligned} \tag{57}$$

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

Next we determine the *initial equal weights*:

$$\text{Weights for measurement Block 1 for } \xi_1 : \quad w_1^{(1)} = 1 \tag{58}$$

$$\text{Weights for measurement Block 2 for } \xi_2 : \quad w_1^{(2)} = \tfrac{1}{2}, \quad w_2^{(2)} = \tfrac{1}{2} \tag{59}$$

Step 1, Block 1: First we compute the *data for* η_1. Using (58) and (55)

$$\begin{aligned} \eta_{1,1} &= \pm w_1^{(1)} \left(x_{1,1}^{(1)} - \overline{x_1^{(1)}} \right) = \pm 1 \cdot (-10) = \mp 10, \\ \eta_{2,1} &= \pm w_1^{(1)} \left(x_{2,1}^{(1)} - \overline{x_1^{(1)}} \right) = \pm 1 \cdot 0 = 0, \\ \eta_{3,1} &= \pm w_1^{(1)} \left(x_{3,1}^{(1)} - \overline{x_1^{(1)}} \right) = \pm 1 \cdot 10 = \pm 10. \end{aligned} \tag{60}$$

We note that here we have no summation as ξ_1 has only one measurement variable and hence there is only one term (in the sum) in the formula for computing the values η_{nq} for η_q.

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

Next we estimate the covariance $\text{Cov}(\eta_1, X_1^{(1)})$ from the the data (60) and (55) in oder to choose the correct sign in (60). We note that $\overline{\eta_1} = 0$.

$$\begin{aligned}
&\widehat{\text{Cov}}(\eta_1, X_1^{(1)}) \\
&= \tfrac{1}{3-1}\left[\eta_{1,1}\left(x_{1,1}^{(1)} - \overline{x_1^{(1)}}\right) + \eta_{2,1}\left(x_{2,1}^{(1)} - \overline{x_1^{(1)}}\right) + \eta_{3,1}\left(x_{3,1}^{(1)} - \overline{x_1^{(1)}}\right)\right] \\
&= \tfrac{1}{2}\big[(\mp 10)\cdot(-10) + 0\cdot 0 + (\pm 10)\cdot(10)\big] = \tfrac{1}{2}\big[\pm 100 + (\pm 100)\big] = \pm 100.
\end{aligned}$$

Hence the estimated correlation is positive if we choose the plus sign in (60), and then we have

$$\eta_{1,1} = -10, \qquad \eta_{2,1} = 0, \qquad \eta_{3,1} = 10. \tag{61}$$

The data (61) of η_1 has already mean $\overline{\eta_1} = 0$ and we compute its standard deviation

$$s_{\eta_1} = \sqrt{\tfrac{1}{2}\left(\eta_{1,1}^2 + \eta_{2,1}^2 + \eta_{3,1}^2\right)} = \sqrt{\tfrac{1}{2}\left((-10)^2 + 0^2 + 10^2\right)} = \sqrt{100} = 10.$$

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

Thus the data of the estimator $\widehat{\xi_1}$ of ξ_1 is given by

$$\begin{aligned}
\xi_{1,1} &= \frac{\eta_{1,1}-\overline{\eta_1}}{s_{\eta_1}} = \frac{-10}{10} = -1, \\
\xi_{2,1} &= \frac{\eta_{2,1}-\overline{\eta_1}}{s_{\eta_1}} = \frac{0}{10} = 0, \\
\xi_{3,1} &= \frac{\eta_{3,1}-\overline{\eta_1}}{s_{\eta_1}} = \frac{10}{10} = 1.
\end{aligned} \tag{62}$$

Step 1, Block 2: First we compute the *data for* η_2. Using (59), (56) and (57), we get

$$\begin{aligned}
\eta_{1,2} &= \pm\left[w_1^{(2)}\left(x_{1,1}^{(2)} - \overline{x_1^{(2)}}\right) + w_2^{(2)}\left(x_{1,2}^{(2)} - \overline{x_2^{(2)}}\right)\right] \\
&= \pm\left[\tfrac{1}{2}\cdot(-5) + \tfrac{1}{2}\cdot(-5)\right] = \mp 5, \qquad (63) \\
\eta_{2,2} &= \pm\left[w_1^{(2)}\left(x_{2,1}^{(2)} - \overline{x_1^{(2)}}\right) + w_2^{(2)}\left(x_{2,2}^{(2)} - \overline{x_2^{(2)}}\right)\right] = \pm\left[\tfrac{1}{2}\cdot 5 + \tfrac{1}{2}\cdot 0\right] = \pm\tfrac{5}{2}, \\
\eta_{3,2} &= \pm\left[w_1^{(2)}\left(x_{3,1}^{(2)} - \overline{x_1^{(2)}}\right) + w_2^{(2)}\left(x_{3,2}^{(2)} - \overline{x_2^{(2)}}\right)\right] = \pm\left[\tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 5\right] = \pm\tfrac{5}{2}.
\end{aligned}$$

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

Using (56), (57), (63) and the facts that $\overline{\eta_2} = 0$ we estimate the covariances $\mathrm{Cov}(\eta_2, X_1^{(2)})$ and $\mathrm{Cov}(\eta_2, X_2^{(2)})$ in order to choose the correct sign in (63).

$$\begin{aligned}
&\widehat{\mathrm{Cov}}(\eta_2, X_1^{(2)}) \\
&= \frac{1}{3-1}\left[\eta_{1,2}\left(x_{1,1}^{(2)} - \overline{x_1^{(2)}}\right) + \eta_{2,2}\left(x_{2,1}^{(2)} - \overline{x_1^{(2)}}\right) + \eta_{3,2}\left(x_{3,1}^{(2)} - \overline{x_1^{(2)}}\right)\right] \\
&= \frac{1}{2}\left[(\mp 5)\cdot(-5) + \left(\pm\frac{5}{2}\right)\cdot 5 + \left(\pm\frac{5}{2}\right)\cdot 0\right] = \pm\frac{75}{4}, \\
&\widehat{\mathrm{Cov}}(\eta_2, X_2^{(2)}) \\
&= \frac{1}{3-1}\left[\eta_{1,2}\left(x_{1,2}^{(2)} - \overline{x_2^{(2)}}\right) + \eta_{2,2}\left(x_{2,2}^{(2)} - \overline{x_2^{(2)}}\right) + \eta_{3,2}\left(x_{3,2}^{(2)} - \overline{x_2^{(2)}}\right)\right] \\
&= \frac{1}{2}\left[(\mp 5)\cdot(-5) + \left(\pm\frac{5}{2}\right)\cdot 0 + \left(\pm\frac{5}{2}\right)\cdot 5\right] = \pm\frac{75}{4}.
\end{aligned}$$

Hence we choose the plus sign in (63) and get the following data for η_2

$$\eta_{1,2} = -5, \qquad \eta_{2,2} = \frac{5}{2}, \qquad \eta_{3,2} = \frac{5}{2}.$$

Ex. 5.1 (a) Step 1 of the Iterative Algorithm

We note that $\overline{\eta_2} = 0$ and estimate the standard deviation of η_2

$$s_{\eta_2} = \sqrt{\tfrac{1}{2}\left(\eta_{1,2}^2 + \eta_{2,2}^2 + \eta_{3,2}^2\right)} = \sqrt{\tfrac{1}{2}\left((-5)^2 + \left(\tfrac{5}{2}\right)^2 + \left(\tfrac{5}{2}\right)^2\right)} = \sqrt{\tfrac{75}{4}} = \tfrac{5\cdot\sqrt{3}}{2}.$$

Thus the data of the estimator $\widehat{\xi_1}$ of ξ_1 is given by

$$\begin{aligned} \xi_{1,2} &= \tfrac{\eta_{1,2} - \overline{\eta_2}}{s_{\eta_2}} = \tfrac{2\cdot(-5)}{5\cdot\sqrt{3}} = -\tfrac{2}{\sqrt{3}}, \\ \xi_{2,2} &= \tfrac{\eta_{2,2} - \overline{\eta_2}}{s_{\eta_2}} = \tfrac{2\cdot(5/2)}{5\cdot\sqrt{3}} = \tfrac{1}{\sqrt{3}}, \\ \xi_{3,2} &= \tfrac{\eta_{3,2} - \overline{\eta_2}}{s_{\eta_2}} = \tfrac{2\cdot(5/2)}{5\cdot\sqrt{3}} = \tfrac{1}{\sqrt{3}}. \end{aligned} \tag{64}$$

For the subsequent steps we summarize the results from (62) and (64):

$$\text{Data for estimator } \widehat{\xi_1} \text{ of } \xi_1\text{:} \quad \xi_{1,1} = -1, \quad \xi_{2,1} = 0, \quad \xi_{3,1} = 1. \tag{65}$$

$$\text{Data for estimator } \widehat{\xi_2} \text{ of } \xi_2\text{:} \quad \xi_{1,2} = -\tfrac{2}{\sqrt{3}}, \quad \xi_{2,2} = \tfrac{1}{\sqrt{3}}, \quad \xi_{3,2} = \tfrac{1}{\sqrt{3}}. \tag{66}$$

Ex. 5.1 (b) Step 2 of the Iterative Algorithm

Using the results from Ex. 5.1 (a) for the structural equation model given in Ex. 5.1 (a), *execute step 2 of the iterative algorithm* with *the centroid weights scheme*.

Solution: *Step 2, Approximation for* ξ_1: The latent variable ξ_1 is only linked to ξ_2. Thus the data for $\rho_1 = e_{1,2}\,\xi_2$ is given by

$$\rho_{n,1} = e_{1,2}\,\xi_{n,2} \qquad \text{with} \qquad e_{1,2} = \text{sign of } \widehat{\mathrm{Cov}}(\widehat{\xi_1}, \widehat{\xi_2}). \tag{67}$$

From (65) and (66) we find

$$\begin{aligned}\widehat{\mathrm{Cov}}(\widehat{\xi_1}, \widehat{\xi_2}) &= \tfrac{1}{3-1}\,[\xi_{1,1}\,\xi_{1,2} + \xi_{2,1}\,\xi_{2,2} + \xi_{3,1}\,\xi_{3,2}] \\ &= \tfrac{1}{2}\left[(-1)\cdot\left(-\tfrac{2}{\sqrt{3}}\right) + 0\cdot\tfrac{1}{\sqrt{3}} + 1\cdot\tfrac{1}{\sqrt{3}}\right] = \tfrac{\sqrt{3}}{2}.\end{aligned} \tag{68}$$

Hence we have $e_{1,2} = 1$, and (67) becomes

$$\rho_{n,1} = e_{1,2}\,\xi_{n,2} = \xi_{n,2}. \tag{69}$$

Ex. 5.1 (b) Step 2 of the Iterative Algorithm

Substituting the data (66) into (69) yields

$$\rho_{1,1} = -\frac{2}{\sqrt{3}}, \qquad \rho_{2,1} = \frac{1}{\sqrt{3}}, \qquad \rho_{3,1} = \frac{1}{\sqrt{3}},$$

and since this data is already standardized we have $\nu_{n,1} = \rho_{n,1}$. Thus,

$$\text{data for } \nu_1: \quad \nu_{1,1} = -\frac{2}{\sqrt{3}}, \qquad \nu_{2,1} = \frac{1}{\sqrt{3}}, \qquad \nu_{3,1} = \frac{1}{\sqrt{3}}. \tag{70}$$

Step 2, Approximation for ξ_2: The latent variable ξ_2 is only linked to ξ_1. Thus the data for $\rho_2 = e_{2,1}\,\xi_1$ is given by

$$\rho_{n,2} = e_{2,1}\,\xi_{n,1} \qquad \text{with} \qquad e_{2,1} = \text{sign of } \widehat{\text{Cov}}(\widehat{\xi}_2, \widehat{\xi}_1). \tag{71}$$

Since $\widehat{\text{Cov}}(\widehat{\xi}_2, \widehat{\xi}_1) = \widehat{\text{Cov}}(\widehat{\xi}_1, \widehat{\xi}_2)$, we have from (68) that $e_{2,1} = 1$, and (71) becomes

$$\rho_{n,2} = e_{2,1}\,\xi_{n,1} = \xi_{n,1}. \tag{72}$$

Ex. 5.1 (b) Step 2 of the Iterative Algorithm

Substituting the data (65) into (72) yields

$$\rho_{1,2} = -1, \qquad \rho_{2,2} = 0, \qquad \rho_{3,2} = 1,$$

and since this data is already standardized we have $\nu_{n,1} = \rho_{n,1}$. Thus,

$$\text{data for } \nu_2: \qquad \nu_{1,2} = -1, \qquad \nu_{2,2} = 0, \qquad \nu_{3,2} = 1. \tag{73}$$

Ex. 5.1 (c) Step 3 (Mode A) of the Iterative Algorithm

Using the results from Ex. 5.1 (a) to (b) for the structural equation model given in Ex. 5.1 (a), *execute step 3 of the iterative algorithm*.

Solution: *New Weights for Bock 1*: The new weight is given by

$$\begin{aligned} w_1^{(1)} &= \widehat{\mathrm{Cov}}(X_1^{(1)}, \nu_1) \\ &= \frac{1}{3-1}\left[\left(x_{1,1}^{(1)} - \overline{x_1^{(1)}}\right)\nu_{1,1} + \left(x_{2,1}^{(1)} - \overline{x_1^{(1)}}\right)\nu_{2,1} + \left(x_{3,1}^{(1)} - \overline{x_1^{(1)}}\right)\nu_{3,1}\right] \\ &= \frac{1}{2}\left[(-10)\cdot\left(-\frac{2}{\sqrt{3}}\right) + 0\cdot\frac{1}{\sqrt{3}} + 10\cdot\frac{1}{\sqrt{3}}\right] = \frac{1}{2}\,\frac{30}{\sqrt{3}} = 5\cdot\sqrt{3}. \end{aligned}$$

where we have used the data (55) for $X_1^{(1)}$ and the data (70) for ν_1.

New Weights for Bock 2: The new weights are given by

$$\begin{aligned} w_1^{(2)} &= \widehat{\mathrm{Cov}}(X_1^{(2)}, \nu_2) \\ &= \frac{1}{3-1}\left[\left(x_{1,1}^{(2)} - \overline{x_1^{(2)}}\right)\nu_{1,2} + \left(x_{2,1}^{(2)} - \overline{x_1^{(2)}}\right)\nu_{2,2} + \left(x_{3,1}^{(2)} - \overline{x_1^{(2)}}\right)\nu_{3,2}\right] \\ &= \frac{1}{2}\left[(-5)\cdot(-1) + 5\cdot 0 + 0\cdot 1\right] = \frac{5}{2} = 2.5, \end{aligned}$$

Ex. 5.1 (c) Step 3 (Mode A) of the Iterative Algorithm

$$\begin{aligned} w_2^{(2)} &= \widehat{\mathrm{Cov}}(X_2^{(2)}, \nu_2) \\ &= \frac{1}{3-1}\left[\left(x_{1,2}^{(2)} - \overline{x_2^{(2)}}\right)\nu_{1,2} + \left(x_{2,2}^{(2)} - \overline{x_2^{(2)}}\right)\nu_{2,2} + \left(x_{3,2}^{(2)} - \overline{x_2^{(2)}}\right)\nu_{3,2}\right] \\ &= \frac{1}{2}\big[(-5)\cdot(-1) + 0\cdot 0 + 5\cdot 1\big] = \frac{10}{2} = 5, \end{aligned}$$

where we have used the data (56) and (57) for $X_1^{(1)}$ and $X_2^{(2)}$, respectively, and the data (73) for ν_2.

Summarizing we have found the following *new weights for the next step 1*:

Weights for measurement Block 1 for ξ_1 : $\quad w_1^{(1)} = 5 \cdot \sqrt{3} \approx 8.66 \qquad (74)$

Weights for measurement Block 2 for ξ_2 : $\quad w_1^{(2)} = \frac{5}{2}, \quad w_2^{(2)} = 5 \qquad (75)$

Ex. 5.1 (d) One More Iterative Step

Using the results from Ex. 5.1 (a) for the structural equation model given in Ex. 5.1 (a) to (c), *execute a second iterative step of the iterative algorithm*.

Solution:

Step 1, Block 1: First we compute the *data for* η_1. Using (74) and (55)

$$
\begin{aligned}
\eta_{1,1} &= \pm w_1^{(1)} \left(x_{1,1}^{(1)} - \overline{x_1^{(1)}} \right) = \pm 5 \cdot \sqrt{3} \cdot (-10) = \mp 50 \cdot \sqrt{3}, \\
\eta_{2,1} &= \pm w_1^{(1)} \left(x_{2,1}^{(1)} - \overline{x_1^{(1)}} \right) = \pm 5 \cdot \sqrt{3} \cdot 0 = 0, \\
\eta_{3,1} &= \pm w_1^{(1)} \left(x_{3,1}^{(1)} - \overline{x_1^{(1)}} \right) = \pm 5 \cdot \sqrt{3} \cdot 10 = \pm 50 \cdot \sqrt{3}.
\end{aligned}
\tag{76}
$$

Next we estimate the covariance $\mathrm{Cov}(\eta_1, X_1^{(1)})$ from the the data (76) and (55) in oder to choose the correct sign in (76). We note that $\overline{\eta_1} = 0$.

$$
\begin{aligned}
&\widehat{\mathrm{Cov}}(\eta_1, X_1^{(1)}) \\
&= \tfrac{1}{3-1}\left[\eta_{1,1}\left(x_{1,1}^{(1)} - \overline{x_1^{(1)}}\right) + \eta_{2,1}\left(x_{2,1}^{(1)} - \overline{x_1^{(1)}}\right) + \eta_{3,1}\left(x_{3,1}^{(1)} - \overline{x_1^{(1)}}\right)\right] \\
&= \tfrac{1}{2}\left[(\mp 50 \cdot \sqrt{3}) \cdot (-10) + 0 \cdot 0 + (\pm 50 \cdot \sqrt{3}) \cdot (10)\right] = \pm 500 \cdot \sqrt{3}.
\end{aligned}
$$

Hence the estimated correlation is positive if we choose the plus sign in (76), and then we have

$$
\eta_{1,1} = -50 \cdot \sqrt{3}, \qquad \eta_{2,1} = 0, \qquad \eta_{3,1} = 50 \cdot \sqrt{3}. \tag{77}
$$

The data (77) of η_1 has already mean $\overline{\eta_1} = 0$ and we compute its standard deviation

$$
\begin{aligned}
s_{\eta_1} &= \sqrt{\tfrac{1}{2}\left(\eta_{1,1}^2 + \eta_{2,1}^2 + \eta_{3,1}^2\right)} \\
&= \sqrt{\tfrac{1}{2}\left((-50 \cdot \sqrt{3})^2 + 0^2 + (50 \cdot \sqrt{3})^2\right)} = \sqrt{75000} = 50 \cdot \sqrt{3}.
\end{aligned}
$$

Thus the data of the estimator $\widehat{\xi_1}$ of ξ_1 is given by

Ex. 5.1 (d) One More Iterative Step

$$\begin{aligned}
\xi_{1,1} &= \frac{\eta_{1,1}-\overline{\eta_1}}{s_{\eta_1}} = \frac{-50\cdot\sqrt{3}}{50\cdot\sqrt{3}} = -1, \\
\xi_{2,1} &= \frac{\eta_{2,1}-\overline{\eta_1}}{s_{\eta_1}} = \frac{0}{50\cdot\sqrt{3}} = 0, \\
\xi_{3,1} &= \frac{\eta_{3,1}-\overline{\eta_1}}{s_{\eta_1}} = \frac{50\cdot\sqrt{3}}{50\cdot\sqrt{3}} = 1.
\end{aligned} \tag{78}$$

Step 1, Block 2: First we compute the *data for* η_2. Using (75), (56) and (57), we get

$$\begin{aligned}
\eta_{1,2} &= \pm\Big[w_1^{(2)}\left(x_{1,1}^{(2)} - \overline{x_1^{(2)}}\right) + w_2^{(2)}\left(x_{1,2}^{(2)} - \overline{x_2^{(2)}}\right)\Big] \\
&= \pm\left[\tfrac{5}{2}\cdot(-5) + 5\cdot(-5)\right] = \mp\tfrac{75}{2}, \qquad (79) \\
\eta_{2,2} &= \pm\Big[w_1^{(2)}\left(x_{2,1}^{(2)} - \overline{x_1^{(2)}}\right) + w_2^{(2)}\left(x_{2,2}^{(2)} - \overline{x_2^{(2)}}\right)\Big] = \pm\left[\tfrac{5}{2}\cdot 5 + 5\cdot 0\right] = \pm\tfrac{25}{2}, \\
\eta_{3,2} &= \pm\Big[w_1^{(2)}\left(x_{3,1}^{(2)} - \overline{x_1^{(2)}}\right) + w_2^{(2)}\left(x_{3,2}^{(2)} - \overline{x_2^{(2)}}\right)\Big] = \pm\left[\tfrac{5}{2}\cdot 0 + 5\cdot 5\right] = \pm 25.
\end{aligned}$$

Ex. 5.1 (d) One More Iterative Step

Using (56), (57), (79) and the facts that $\overline{\eta_2} = 0$ we estimate the covariances $\mathrm{Cov}(\eta_2, X_1^{(2)})$ and $\mathrm{Cov}(\eta_2, X_2^{(2)})$ in order to choose the correct sign in (79).

$$\begin{aligned}
&\widehat{\mathrm{Cov}}(\eta_2, X_1^{(2)}) \\
&= \tfrac{1}{3-1}\left[\eta_{1,2}\left(x_{1,1}^{(2)} - \overline{x_1^{(2)}}\right) + \eta_{2,2}\left(x_{2,1}^{(2)} - \overline{x_1^{(2)}}\right) + \eta_{3,2}\left(x_{3,1}^{(2)} - \overline{x_1^{(2)}}\right)\right] \\
&= \tfrac{1}{2}\left[\left(\mp\tfrac{75}{2}\right)\cdot(-5) + \left(\pm\tfrac{25}{2}\right)\cdot 5 + (\pm 5)\cdot 0\right] = \pm\tfrac{500}{4} = \pm 125, \\
&\widehat{\mathrm{Cov}}(\eta_2, X_2^{(2)}) \\
&= \tfrac{1}{3-1}\left[\eta_{1,2}\left(x_{1,2}^{(2)} - \overline{x_2^{(2)}}\right) + \eta_{2,2}\left(x_{2,2}^{(2)} - \overline{x_2^{(2)}}\right) + \eta_{3,2}\left(x_{3,2}^{(2)} - \overline{x_2^{(2)}}\right)\right] \\
&= \tfrac{1}{2}\left[\left(\mp\tfrac{75}{2}\right)\cdot(-5) + \left(\pm\tfrac{25}{2}\right)\cdot 0 + (\pm 5)\cdot 5\right] = \pm\tfrac{425}{4}.
\end{aligned}$$

Hence we choose the plus sign in (79) and get the following data for η_2

$$\eta_{1,2} = -\tfrac{75}{2}, \qquad \eta_{2,2} = \tfrac{25}{2}, \qquad \eta_{3,2} = 5.$$

Ex. 5.1 (d) One More Iterative Step

We note that $\overline{\eta_2} = 0$ and estimate the standard deviation of η_2

$$s_{\eta_2} = \sqrt{\frac{1}{2}\left(\eta_{1,2}^2 + \eta_{2,2}^2 + \eta_{3,2}^2\right)} = \sqrt{\frac{1}{2}\left(\left(-\frac{75}{2}\right)^2 + \left(\frac{25}{2}\right)^2 + 5^2\right)} = \frac{5\cdot\sqrt{127}}{2}.$$

Thus the data of the estimator $\widehat{\xi_1}$ of ξ_1 is given by

$$\begin{aligned} \xi_{1,2} &= \frac{\eta_{1,2}-\overline{\eta_2}}{s_{\eta_2}} = \frac{2}{5\cdot\sqrt{127}} \cdot \left(-\frac{75}{2}\right) = -\frac{15}{\sqrt{127}}, \\ \xi_{2,2} &= \frac{\eta_{2,2}-\overline{\eta_2}}{s_{\eta_2}} = \frac{2}{5\cdot\sqrt{127}} \cdot \frac{25}{2} = \frac{5}{\sqrt{127}}, \\ \xi_{3,2} &= \frac{\eta_{3,2}-\overline{\eta_2}}{s_{\eta_2}} = \frac{2\cdot 5}{5\cdot\sqrt{127}} = \frac{2}{\sqrt{127}}. \end{aligned} \tag{80}$$

For the subsequent steps we summarize the results from (78) and (80):

Data for estimator $\widehat{\xi_1}$ of ξ_1: $\quad \xi_{1,1} = -1, \quad \xi_{2,1} = 0, \quad \xi_{3,1} = 1.$ (81)

Data for estimator $\widehat{\xi_2}$ of ξ_2: $\xi_{1,2} = -\frac{15}{\sqrt{127}}, \ \xi_{2,2} = \frac{5}{\sqrt{127}}, \ \xi_{3,2} = \frac{2}{\sqrt{127}}.$ (82)

Ex. 5.1 (d) One More Iterative Step

Step 2, Approximation for ξ_1: The latent variable ξ_1 is only linked to ξ_2. Thus the data for $\rho_1 = e_{1,2}\,\xi_2$ is given by

$$\rho_{n,1} = e_{1,2}\,\xi_{n,2} \qquad \text{with} \qquad e_{1,2} = \text{sign of } \widehat{\text{Cov}}(\widehat{\xi_1}, \widehat{\xi_2}). \tag{83}$$

From (81) and (82) we find

$$\begin{aligned} \widehat{\text{Cov}}(\widehat{\xi_1}, \widehat{\xi_2}) &= \tfrac{1}{3-1}\,[\xi_{1,1}\,\xi_{1,2} + \xi_{2,1}\,\xi_{2,2} + \xi_{3,1}\,\xi_{3,2}] \\ &= \tfrac{1}{2}\left[(-1)\cdot\left(-\tfrac{15}{\sqrt{127}}\right) + 0\cdot\tfrac{5}{\sqrt{127}} + 1\cdot\tfrac{2}{\sqrt{127}}\right] = \tfrac{17}{2\cdot\sqrt{127}}. \end{aligned} \tag{84}$$

Hence we have $e_{1,2} = 1$, and (83) becomes

$$\rho_{n,1} = e_{1,2}\,\xi_{n,2} = \xi_{n,2}. \tag{85}$$

Substituting the data (82) into (85) yields

$$\rho_{1,1} = -\frac{15}{\sqrt{127}}, \qquad \rho_{2,1} = \frac{5}{\sqrt{127}}, \qquad \rho_{3,1} = \frac{2}{\sqrt{127}},$$

and since this data is already standardized we have $\nu_{n,1} = \rho_{n,1}$. Hence,

$$\text{data for } \nu_1: \quad \nu_{1,1} = -\frac{15}{\sqrt{127}}, \quad \nu_{2,1} = \frac{5}{\sqrt{127}}, \quad \nu_{3,1} = \frac{2}{\sqrt{127}}. \tag{86}$$

Step 2, Approximation for ξ_2: The latent variable ξ_2 is only linked to ξ_1. Thus the data for $\rho_2 = e_{2,1}\,\xi_1$ is given by

$$\rho_{n,2} = e_{2,1}\,\xi_{n,1} \qquad \text{with} \qquad e_{2,1} = \text{sign of } \widehat{\text{Cov}}(\widehat{\xi_2}, \widehat{\xi_1}). \tag{87}$$

Since $\widehat{\text{Cov}}(\widehat{\xi_2}, \widehat{\xi_1}) = \widehat{\text{Cov}}(\widehat{\xi_1}, \widehat{\xi_2})$, we have from (84) that $e_{2,1} = 1$, and (87) becomes

$$\rho_{n,2} = e_{2,1}\,\xi_{n,1} = \xi_{n,1}. \tag{88}$$

Substituting the data (81) into (88) yields

$$\rho_{1,2} = -1, \qquad \rho_{2,2} = 0, \qquad \rho_{3,2} = 1,$$

Ex. 5.1 (d) One More Iterative Step

and since this data is already standardized we have $\nu_{n,1} = \rho_{n,1}$. Hence,

$$\text{data for } \nu_2: \qquad \nu_{1,2} = -1, \qquad \nu_{2,2} = 0, \qquad \nu_{3,2} = 1. \tag{89}$$

New Weights for Bock 1: The new weight is given by

$$\begin{aligned} w_1^{(1)} &= \widehat{\text{Cov}}(X_1^{(1)}, \nu_1) \\ &= \frac{1}{3-1}\left[\left(x_{1,1}^{(1)} - \overline{x_1^{(1)}}\right)\nu_{1,1} + \left(x_{2,1}^{(1)} - \overline{x_1^{(1)}}\right)\nu_{2,1} + \left(x_{3,1}^{(1)} - \overline{x_1^{(1)}}\right)\nu_{3,1}\right] \\ &= \frac{1}{2}\left[(-10)\cdot\left(-\frac{15}{\sqrt{127}}\right) + 0\cdot\frac{5}{\sqrt{127}} + 10\cdot\frac{2}{\sqrt{127}}\right] = \frac{1}{2}\,\frac{170}{\sqrt{127}} = \frac{85}{\sqrt{127}} \approx 7.54. \end{aligned}$$

where we have used the data (55) for $X_1^{(1)}$ and the data (86) for ν_1.

New Weights for Bock 2: The new weights are given by

$$\begin{aligned} w_1^{(2)} &= \widehat{\text{Cov}}(X_1^{(2)}, \nu_2) \\ &= \frac{1}{3-1}\left[\left(x_{1,1}^{(2)} - \overline{x_1^{(2)}}\right)\nu_{1,2} + \left(x_{2,1}^{(2)} - \overline{x_1^{(2)}}\right)\nu_{2,2} + \left(x_{3,1}^{(2)} - \overline{x_1^{(2)}}\right)\nu_{3,2}\right] \end{aligned}$$

Ex. 5.1 (d) One More Iterative Step

$$w_1^{(2)} = \widehat{\text{Cov}}(X_1^{(2)}, \nu_2) = \tfrac{1}{2}\big[(-5)\cdot(-1) + 5\cdot 0 + 0\cdot 1\big] = \tfrac{5}{2} = 2.5,$$

$$\begin{aligned} w_2^{(2)} &= \widehat{\text{Cov}}(X_2^{(2)}, \nu_2) \\ &= \tfrac{1}{3-1}\left[\left(x_{1,2}^{(2)} - \overline{x_2^{(2)}}\right)\nu_{1,2} + \left(x_{2,2}^{(2)} - \overline{x_2^{(2)}}\right)\nu_{2,2} + \left(x_{3,2}^{(2)} - \overline{x_2^{(2)}}\right)\nu_{3,2}\right] \\ &= \tfrac{1}{2}\big[(-5)\cdot(-1) + 0\cdot 0 + 5\cdot 1\big] = \tfrac{10}{2} = 5, \end{aligned}$$

where we have used the data (56) and (57) for $X_1^{(1)}$ and $X_2^{(2)}$, respectively, and the data (89) for ν_2.

Summarizing we have found the following *new weights for the next step 1*:

$$\text{Weights for measurement Block 1 for } \xi_1 : \quad w_1^{(1)} = \tfrac{85}{\sqrt{127}} \approx 7.54 \qquad (90)$$

$$\text{Weights for measurement Block 2 for } \xi_2 : \quad w_1^{(2)} = \tfrac{5}{2}, \quad w_2^{(2)} = 5 \qquad (91)$$

Ex. 5.1 (e) Further Iterative Steps

Inspect the results and computations from Ex. 5.1 (a) to (d) for the structural equation model given in Ex. 5.1 (a), and in particular compare the weights $w_p^{(q)}$ from the two iterative steps and observe their effect. *Use your observations to predict the results of subsequent iterative steps. What happens after the third iterative step?*

Solution: We start by comparing the weights computed in the first and second iterative step: In both iterative steps we had (see (75) and (91))

$$w_1^{(2)} = \tfrac{5}{2} \qquad \text{and} \qquad w_2^{(2)} = 5, \tag{92}$$

i.e. the weights for measurement block 2 have not changed. For the weight $w_1^{(1)}$ of measurement block 1 we had different values in the two iterative steps. In the first step we had $w_1^{(1)} = 5 \cdot \sqrt{3} \approx 8.66$ (see (74)), and in the second step we found (see (90))

$$w_1^{(1)} = \tfrac{85}{\sqrt{127}} \approx 7.54. \tag{93}$$

Ex. 5.1 (e) Further Iterative Steps

Next we inspect the computation done in the steps of the algorithm and consider what will happen in the third iterative step:

Step 1, Block 1: Here we compute first

$$\eta_{n,1} = \pm w_1^{(1)} \left(x_{n,1}^{(1)} - \overline{x_1^{(1)}} \right), \tag{94}$$

where the sign has to be chosen such that $\widehat{\mathrm{Cov}}(\eta_1, X_1^{(1)})$ is positive, and afterwards we standardize the data for η_1 to obtain data for $\widehat{\xi_1}$.

So far all our values for $w_1^{(1)}$ have been positive We note that in (94), if $w_1^{(1)} > 0$ and if we choose the plus sign, the data for η_1 is just a positive multiple of the data for $X_1^{(1)}$. Thus if $w_1^{(1)} > 0$ and if we choose the plus sign, $\widehat{\mathrm{Cov}}(\eta_1, X_1^{(1)})$ will have the same sign as

$$\widehat{\mathrm{Cov}}(X_1^{(1)}, X_1^{(1)}) = \mathrm{Var}(X_1^{(1)}) > 0.$$

Hence for positive weights $w_1^{(1)}$ we must choose the plus sign in (94), and we get

$$\eta_{n,1} = w_1^{(1)} \left(x_{n,1}^{(1)} - \overline{x_1^{(1)}}\right), \tag{95}$$

Next we note that, because the data (95) for η_1 is obtained by multiplying the centered data of $X_1^{(1)}$ with a positive factor, standardizing the data for η_1 will give the same result as standardizing the data for $X_1^{(1)}$. Hence for positive weights $w_1^{(1)}$ the data for $\widehat{\xi_1}$ does not depend on the value of the positive weight $w_1^{(1)}$ and we get always the same result as in the first and second step (see (65) and (81)), namely

$$\xi_{1,1} = -1, \qquad \xi_{2,1} = 0, \qquad \xi_{3,1} = 1. \tag{96}$$

In particular we will get (96) in the third iterative step.

Step 1, Block 2: Here we first compute

$$\eta_{n,2} = \pm\left[w_1^{(2)} \left(x_{n,1}^{(2)} - \overline{x_1^{(2)}}\right) + w_2^{(2)} \left(x_{n,2}^{(2)} - \overline{x_2^{(2)}}\right)\right], \tag{97}$$

Ex. 5.1 (e) Further Iterative Steps

where the sign has to be chosen such that $\widehat{\text{Cov}}(\eta_2, X_1^{(2)})$, $\widehat{\text{Cov}}(\eta_2, X_2^{(2)})$ or both are positive, and afterwards we standardize the data for η_2 to obtain data for $\widehat{\xi}_2$.

As the weights $w_1^{(2)}$ and $w_2^{(2)}$ are still the same as in the previous step, we will also get the same results for the data for $\widehat{\xi}_2$ as in the previous step, namely in the third iterative step we get (see (82))

$$\xi_{1,2} = -\frac{15}{\sqrt{127}}, \qquad \xi_{2,2} = \frac{5}{\sqrt{127}}, \qquad \xi_{3,2} = \frac{2}{\sqrt{127}}. \tag{98}$$

So in the third iterative step we have found *exactly the same values for the data of $\widehat{\xi}_1$ and $\widehat{\xi}_2$ as in the second iterative step*.

Step 2, Approximation of ξ_1 and ξ_2 in the inner model: Here we have two identical weights from the centroid weighting scheme given by

$$e_{1,2} = e_{2,1} = \text{sign of } \widehat{\text{Cov}}(\widehat{\xi}_1, \widehat{\xi}_2)$$

Ex. 5.1 (e) Further Iterative Steps

As the data for $\widehat{\xi_1}$ and $\widehat{\xi_2}$ in the third iterative step is the same same as in the previous iterative step, their empirical covariance is also the same and we find (see (84))

$$e_{1,2} = e_{2,1} = 1.$$

Thus the data for ρ_1 and ρ_2 is given by the same formulas as in the second iterative step and we have (see (85) and (88))

$$\rho_{n,1} = \xi_{n,2} \qquad \text{and} \qquad \rho_{n,2} = \xi_{n,1} \tag{99}$$

As the data of $\widehat{\xi_1}$ and $\widehat{\xi_2}$ was already standardized, (99) immediately implies for the third iterative step

$$\nu_{n,1} = \xi_{n,2} \qquad \text{and} \qquad \nu_{n,2} = \xi_{n,1}$$

which is just the same formula as in the second iterative step.

As we found in step 1 that the data of $\widehat{\xi_1}$ and $\widehat{\xi_2}$ in the third step has the same values as in the second step, *the data for ν_1 and ν_2 in both steps has also the same values* and we find (from (98) and (96))

Ex. 5.1 (e) Further Iterative Steps

$$\nu_{1,1} = -\tfrac{15}{\sqrt{127}}, \qquad \nu_{2,1} = \tfrac{5}{\sqrt{127}}, \qquad \nu_{3,1} = \tfrac{2}{\sqrt{127}}, \tag{100}$$

$$\nu_{1,2} = -1, \qquad \nu_{2,2} = 0, \qquad \nu_{3,2} = 1. \tag{101}$$

(See (86) and (89) in the second iterative step for comparison.)

Step 3, computation of the new weights: The new weights are computed with the formulas

$$w_1^{(1)} = \widehat{\mathrm{Cov}}(X_1^{(1)}, \nu_1), \qquad w_1^{(2)} = \widehat{\mathrm{Cov}}(X_1^{(2)}, \nu_2), \qquad w_2^{(2)} = \widehat{\mathrm{Cov}}(X_2^{(2)}, \nu_2),$$

and as we have the same data (100) and (101) for ν_1 and ν_2 in the third and second iterative step, we will also get the same weights as in the second step. Hence, we find (see (90) and (91))

$$w_1^{(1)} = \tfrac{85}{\sqrt{127}} \approx 7.54, \qquad w_1^{(2)} = \tfrac{5}{2} \qquad \text{and} \qquad w_2^{(2)} = 5. \tag{102}$$

Ex. 5.1 (e) Further Iterative Steps

Further Iterative Steps: As weights at the beginning of the forth iterative step (see (102)) are the same as the weights at the beginning of the third iterative step (see (92) and (93)), it is clear that any further iterative steps will produce exactly the same results as the second and the third iterative step.

What happens after the third iterative step? After each step we have to test the stopping criterion, and in the third step we get (for the first time) the *same weights as in the previous step*. Computing the stopping criterion after step 3, we therefore find

$$\Delta = \max \left\{ \begin{array}{c} \left|(w_1^{(1)})^{\text{new}} - (w_1^{(1)})^{\text{old}}\right|, \\ \left|(w_1^{(2)})^{\text{new}} - (w_1^{(2)})^{\text{old}}\right|, \\ \left|(w_2^{(2)})^{\text{new}} - (w_2^{(2)})^{\text{old}}\right| \end{array} \right\} = 0$$

and *the iterative algorithm stops*.

Ex. 5.1 (f) Values of the Latent Variables and Path Coeffs.

Using the results from Ex. 5.1 (a) to (e) for the structural equation model given in Ex. 5.1 (a), *stop the iterative algorithm after the third step* and *compute the estimates of the latent variables and the path coefficients*. Inspect your results.

<u>Solution:</u>

Final Values for the Latent Variables: From the considerations in Ex. 5.1 (e) we know that the values of another application of step 1 (after the end of the third iterative step) provide the following final values for $\widehat{\xi_1}$ and $\widehat{\xi_2}$ (see (96) and (98)):

$$\xi_{1,1} = -1, \qquad \xi_{2,1} = 0, \qquad \xi_{3,1} = 1, \tag{103}$$

$$\xi_{1,2} = -\frac{15}{\sqrt{127}}, \qquad \xi_{2,2} = \frac{5}{\sqrt{127}}, \qquad \xi_{3,2} = \frac{2}{\sqrt{127}}. \tag{104}$$

Computation of the Path Coefficients: Here we have to compute only one path coefficient $\beta_{2,1}$ (for the arrow pointing from ξ_1 to ξ_2).

Ex. 5.1 (f) Values of the Latent Variables and Path Coeffs.

The matrix $\boldsymbol{\Xi}_2$ contains here only the data for the final values of the one variable ξ_1 (as this is the only variable from whom an arrow points to ξ_2). Thus (using (103))

$$\boldsymbol{\Xi}_2 = \begin{pmatrix} \xi_{1,1} \\ \xi_{2,1} \\ \xi_{3,1} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

and the vector $\boldsymbol{\xi}_2$ contains the final values for ξ_2 and is given by (use (104))

$$\boldsymbol{\xi}_2 = \begin{pmatrix} \xi_{1,2} \\ \xi_{2,2} \\ \xi_{3,2} \end{pmatrix} = \begin{pmatrix} -\frac{15}{\sqrt{127}} \\ \frac{5}{\sqrt{127}} \\ \frac{2}{\sqrt{127}} \end{pmatrix}.$$

The coefficient $\beta_{2,1}$ is computed with the least squares formula

$$\beta_{2,1} = (\boldsymbol{\Xi}_2' \, \boldsymbol{\Xi}_2)^{-1} \boldsymbol{\Xi}_2' \, \boldsymbol{\xi}_2. \tag{105}$$

Ex. 5.1 (f) Values of the Latent Variables and Path Coeffs.

We start by computing $(\boldsymbol{\Xi}_2' \, \boldsymbol{\Xi}_2)^{-1}$,

$$\boldsymbol{\Xi}_2' \, \boldsymbol{\Xi}_2 = (-1, 0, 1) \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = (-1)^2 + 0^2 + 1^2 = 2$$

and thus

$$(\boldsymbol{\Xi}_2' \, \boldsymbol{\Xi}_2)^{-1} = 2^{-1} = \tfrac{1}{2}. \tag{106}$$

Next we compute

$$\boldsymbol{\Xi}_2' \, \boldsymbol{\xi}_2 = (-1, 0, 1) \begin{pmatrix} -\frac{15}{\sqrt{127}} \\ \frac{5}{\sqrt{127}} \\ \frac{2}{\sqrt{127}} \end{pmatrix} = \frac{15}{\sqrt{127}} + \frac{2}{\sqrt{127}} = \frac{17}{\sqrt{127}} \approx 1.51. \tag{107}$$

Substituting (107) and (106) into (105) yields

$$\beta_{2,1} = (\boldsymbol{\Xi}_2' \, \boldsymbol{\Xi}_2)^{-1} \boldsymbol{\Xi}_2' \, \boldsymbol{\xi}_2 = \tfrac{1}{2} \cdot \tfrac{17}{\sqrt{127}} = \tfrac{17}{2 \cdot \sqrt{127}} \approx 0.754.$$

Ex. 5.2: Comparing the PLS Model and the LISREL Model

Compare the coefficients of the PLS model from Ex. 5.1 with the LISREL model from Ex. 4.3. To do this, you need to consider the *standardized coefficients*, because the variables in the two models are scaled differently.

For a *regression equation*

$$X - \mu_X = \gamma_1 \, \xi_1 + \gamma_2 \, \xi_2 + \ldots + \gamma_m \, \xi_m + \delta,$$

where δ is the error term and $\gamma_1, \gamma_2, \ldots, \gamma_m$ the coefficients, the *standardized coefficients* $\widetilde{\gamma}_j$ are given by

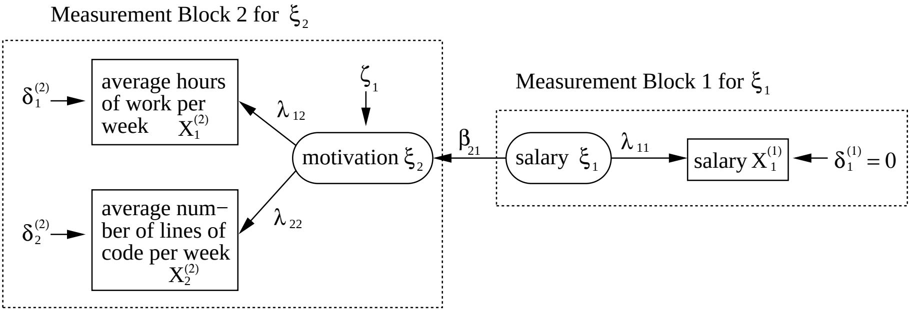
$$\widetilde{\gamma}_j = \gamma_j \cdot \frac{\sigma_{\xi_j}}{\sigma_X} = \gamma_j \cdot \frac{\text{standard deviation of } \xi_j}{\text{standard deviation of } X}. \qquad (108)$$

For computing the standardized coefficients, we have to *estimate the standard deviations in (108) by the empirical standard deviations* s_{ξ_j} *and* s_X computed from the data.

Ex. 5.2: Comparing the PLS Model and the LISREL Model

Solution: For the *PLS model* (diagram below) we found $\beta_{2,1} \approx 0.754$. To standardize $\beta_{2,1}$ we need the standard deviations for the final data of ξ_1 and ξ_2. As ξ_1 and ξ_2 are standardized we have $\sigma_{\xi_1} = s_{\xi_1} = 1$ and $\sigma_{\xi_2} = s_{\xi_2} = 1$, and the standardized coefficient for $\beta_{2,1}$ is

$$\widetilde{\beta_{2,1}} = \beta_{2,1} \cdot \frac{s_{\xi_1}}{s_{\xi_2}} = \beta_{2,1} \approx 0.745.$$



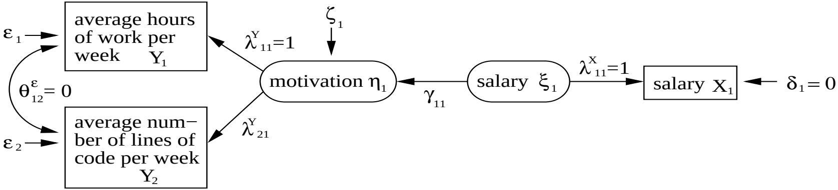
Ex. 5.2: Comparing the PLS Model and the LISREL Model

For the *LISREL model* (diagram below) we found that $\gamma_{1,1} = 0.25 = 1/4$, and we computed $\mathrm{Var}(\xi_1) = \phi_{1,1} = 100$. Hence the empirical standard deviation $s_{\xi_1} = 10$. The variance of η_1 was no directly computed, but from (40), (51) and $\lambda^Y_{2,1} = 2$ we have

$$\mathrm{Var}(\eta_1) = \frac{\mathrm{Cov}(Y_1, Y_2)}{\lambda^Y_{2,1}} = \frac{12.5}{2} = 6.25,$$

and so $s_{\eta_1} = \sqrt{6.25} = 5/2$. Hence the standardized coefficient for $\gamma_{1,1}$ is

$$\widetilde{\gamma_{1,1}} = \gamma_{1,1} \cdot \frac{s_{\xi_1}}{s_{\eta_1}} = \frac{1}{4} \cdot \frac{10}{5/2} = 1.$$



Ex. 5.2: Comparing the PLS Model and the LISREL Model

The standardized coefficients for the path from ξ_1 to ξ_2 (PLS) and ξ to η_1 (LISREL), respectively, in the inner structural model are $\widetilde{\beta_{2,1}} \approx 0.745$ (PLS) and $\widetilde{\gamma_{1,1}} = 1$ (LISREL). So we note the two models give slightly different results for our example.