The University of Sussex – Department of Mathematics

Wavelets and Data Compression

G1025 & G5080

Lecture Notes – Autumn Term 2010

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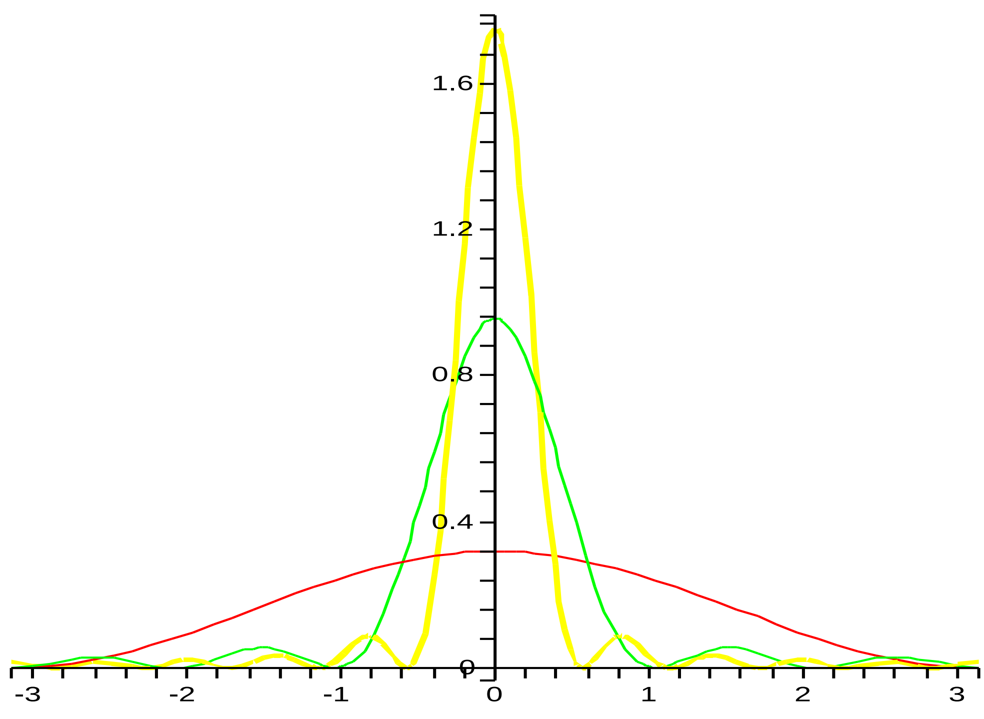

Figure 1: Fejér's kernel K_m for $m = 1, 5, 10$.

Lecture notes and course material by Alex Sobolev, Holger Wendland, and others, who previously taught the course 'Wavelets and Data Compression' at the University of Sussex, served as a starting point for the current lecture notes. The current lecture notes are more than twice as many pages as the previous version. Apart from corrections and improvements, many new examples and exercises have been added compared to the previous lecture notes.

Contents

Introduction

Let us consider the following problem. Assume, we want to **store a moderately sized image** with a resolution of 1024×1024 pixels (this corresponds to a one mega pixel camera!!). For each pixel our image has the colour information in RGB format with 8 bits per colour, which gives the nowadays usual 24 bits colour depth. If we simply store all 3 Bytes for each pixel we would end up with 3 MB storage requirement. This is already quite a lot for such a small image. Modern cameras have a resolution of up to 7 MB such that storing an image in an uncompressed way would lead to 21 MB memory requirement. Obviously, this is not acceptable. Hence, it is necessary to find better representations of the image, meaning also compression techniques. It is the goal of this lecture to give an introduction into this field. In particular into the area of compression using the **Fourier (or cosine) transform** and **wavelets**. Both have been already developed to an industry standard:

- The FDCT (fast discrete cosine transform) is the basis of the classical JPEG standard.
- The FDWT (fast discrete wavelet transform) is the basis of the new JPEG 2000 standard.

However, we will only discuss the mathematical ideas here, leaving out most details on efficient implementation.

The idea behind both methods is the following one. Let us assume that we can treat each colour component of the image separately. Then, we can interpret the colour distribution as discrete values of a continuous function, i.e. we assume that there exists a function

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad \text{with} \qquad f(i,j) = c_{i,j}, \quad i,j \in \{1,2,\ldots,1024\}.$$

It is now our goal to find a 'better' representation of f in the form

$$f(x,y) = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij}\, e_{ij}(x,y) \tag{0.0.1}$$

with certain given 'basis' functions e_{ij} and coefficients a_{ij} that need to be determined. We will encounter different basis functions and strategies to compute the coefficients. Though the sum in (0.0.1) is bi-infinite, which makes an efficient evaluation of f rather difficult, it often is actually finite or the coefficients decay so far that it can be made finite while making only a small error in doing so.

For now, it suffices to see that representing the image in such a way consists of two steps:

- **Coding**: This step has to be done only once and hence might consume some time. In this step the coefficients a_{ij} are computed, filtered, for example by setting all coefficients to zero which are smaller than a given threshold, and finally stored. For storing the remaining coefficients it is important also to store their location, that is, their indices. This has to be done efficiently, because otherwise the gain of a 'sparse' representation is lost again. This means that what is stored in an efficient way can be quickly retrieved and allows us to reconstruct a **good approximation** of the original image.
- **Decoding**: This has to be done each time the image is viewed. Hence, this step has to be performed in real time. It consists of recovering the (approximations of the) colour values c_{ij} as the function values $f(i,j)$.

In this lecture, we will often make additional assumptions on the function f. First of all, we will consider univariate functions (that is, functions of one variable) $f : \mathbb{R} \to \mathbb{R}$ instead of bivariate functions. Second, the function will often be considered to be periodic with period 2π, that is,

$$f(x + 2\pi) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

This is not as limiting as it seems, since wavelets are used frequently in **signal compression** and **signal analysis**. For example, acoustic signals (music, speech, bird voices) as a function of time are examples of univariate functions and these can stored in a compressed format with the help of wavelets.

Chapter 1

Revision: Linear Spaces

This chapter reviews concepts that are familiar from linear algebra. First we revise the definition of a **linear space** and a **subspace** and consider some examples in Section 1.1. Then we review the concepts of **linear independence** and a **basis** in Section 1.2.

1.1 Linear Spaces and Subspaces

In this lecture course, the considered fields $\mathbb{K}$ are either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.1 (linear space or vector space)
A ***linear space*** *(or* ***vector space****) over a field* $\mathbb{K}$ *is a non-empty set* X *with two algebraic operations, namely* ***vector addition*** $\oplus : X \times X \to X$ *and* ***scalar multiplication*** $\odot : \mathbb{K} \times X \to X$*, such that the following properties are all satisfied:*

(i) $x \oplus y = y \oplus x$ *for all* $x, y \in X$*, that is, addition is* ***commutative****.*

(ii) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ *for all* $x, y, z \in X$*, that is, addition is* ***associative****.*

(iii) There exists a unique vector $\mathcal{O}$ *such that* $x \oplus \mathcal{O} = x$ *for every* $x \in X$*. The vector* $\mathcal{O}$ *is called the* ***zero vector****.*

(iv) For every $x \in X$ *there exists a unique vector, denoted* $-x$*, such that* $x \oplus (-x) = 0$*. The vector* $-x$ *is called the* ***(additive) inverse*** *of* x*.*

(v) $\alpha \odot (\beta \odot x) = (\alpha\,\beta) \odot x$ *for all* $\alpha, \beta \in \mathbb{K}$ *and all* $x \in X$*;*

(vi) $1 \odot x = x$ *for all* $x \in X$ *(where* 1 *is the number* 1 *in* $\mathbb{R}$ *and* $\mathbb{C}$*);*

(vii) $\alpha \odot (x \oplus y) = (\alpha \odot x) \oplus (\alpha \odot y)$ *for all* $\alpha \in \mathbb{K}$ *and all* $x, y \in X$ *(1st distributive law);*

(viii) $(\alpha + \beta) \odot x = (\alpha \odot x) \oplus (\beta \odot x)$ *for all* $\alpha, \beta \in \mathbb{K}$ *and all* $x \in X$ *(2nd distributive law).*

If $\mathbb{K} = \mathbb{R}$*, the set* X *is called a* ***real linear space****. If* $\mathbb{K} = \mathbb{C}$*, the set* X *is called a* ***complex linear space****.*

Here are some examples of vector spaces.

Example 1.2 (elementary examples of linear spaces)
The scalar field itself and tensor products of it are vector spaces:

(a) The complex numbers $\mathbb{C}$ and the d-dimensional complex space $\mathbb{C}^d$ are complex linear spaces.

(b) The real numbers $\mathbb{R}$ and the d-dimensional Euclidean space $\mathbb{R}^d$ are real linear spaces. □

Example 1.3 (space $\ell(\mathbb{N})$ of infinite sequences)
Let $\ell(\mathbb{N})$ be the space of all infinite sequences of the form

$$x = (x_1, x_2, \ldots, x_k, \ldots) = (x_k)_{k \in \mathbb{N}},$$

with the elements $x_k \in \mathbb{K}$. Addition and multiplication by scalars is defined element-wise:

$$\begin{aligned} (x+y)_k &:= x_k + y_k, \qquad k \in \mathbb{N}; \\ (\alpha\, x)_k &:= \alpha\, x_k, \qquad k \in \mathbb{N}, \quad \alpha \in \mathbb{K}. \end{aligned}$$

It is easy to see that X with these operations satisfies all the requirements of a vector space. □

Exercise 1 *Show that the set $\ell(\mathbb{N})$ of infinite sequences introduced in Example 1.3 with the given addition and scalar multiplication is a vector space.*

Example 1.4 (space of continuous complex-valued functions on $[a, b]$)
The set of continuous complex-valued functions $C([a, b])$ on the interval $[a, b]$ forms a complex linear space (vector space) with the **pointwise addition**

$$(f + g)(x) := f(x) + g(x) \qquad \text{for all } x \in [a, b] \tag{1.1.1}$$

and the **pointwise scalar multiplication**

$$(\alpha\, f)(x) := \alpha\, f(x) \qquad \text{for all } x \in [a, b] \text{ and all } \alpha \in \mathbb{C}. \tag{1.1.2}$$

This is verified in Exercise 2 below. □

Exercise 2 *Show that the set $C([a, b])$ of continuous complex-valued functions on $[a, b]$ with the pointwise addition (1.1.1) and the pointwise scalar multiplication (1.1.2) is a complex linear space.*

Exercise 3 *Show that the set $\Pi(\mathbb{R})$ of real-valued polynomials on $\mathbb{R}$ with the pointwise addition (1.1.1) and the pointwise scalar multiplication (1.1.2) forms a real linear space.*

Often we will work with subspaces of (larger) vector spaces.

Definition 1.5 (subspace of a linear space)
A ***subspace*** *of a linear space X over the field $\mathbb{K}$ is a non-empty subset $Y \subset X$ such that for all $y, w \in Y$ and all scalars $\alpha, \beta \in \mathbb{K}$ one has $\alpha\, y + \beta\, w \in Y$.*

A subspace of a linear space is (as the name implies) also a linear space.

Lemma 1.6 (subspace of a linear space is also a linear space)
Let X be a linear space, and let Y be a subspace of X. Then Y (with the same addition and same scalar multiplication as X) is also a vector space (over the same field $\mathbb{K}$).

Example 1.7 (real line through origin in $\mathbb{R}^2$)
Let $\mathbb{R}^2$ be the usual 2-dimensional Euclidean space which is a real linear space. Consider an arbitrary vector $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$. Then $Y := \{\alpha\,\mathbf{x} \,:\, \alpha \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$. □

Example 1.8 (function spaces on $[a, b]$)
Let $F([a, b])$ denote the space of all complex-valued functions on $[a, b]$ with the pointwise addition (1.1.1) and the pointwise scalar multiplication (1.1.2). Then $F([a, b])$ is a complex linear space, and the space $C([a, b])$ of continuous complex-valued functions is a subspace. □

Exercise 4 *For each of the following given linear spaces X and their subsets Y, investigate whether the subset Y is a subspace. Give proofs of your answers! (You do* **not** *have to verify that the given linear space X is a linear space.)*

(a) *Is the subset $Y = \{\mathbf{y} + \alpha\,\mathbf{x} \,:\, \alpha \in \mathbb{R}\}$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ are two arbitrary vectors, a subspace of the Euclidean space $X = \mathbb{R}^2$?*

(b) *Let $X = C([a, b])$ be the space of continuous complex-valued functions on $[a, b]$. Is the set Y of constant functions on $[a, b]$,*

$$Y = \{f : [a, b] \to \mathbb{R},\ f(x) := c \text{ for all } x \in [a, b] \,:\, c \in \mathbb{C}\}$$

a subspace of $X = C([a, b])$?

(c) *Let $X = \mathbb{C}$ be the usual complex linear space of complex numbers. Is the subset $Y = \mathbb{R}$ a subspace?*

(d) *Is the set $Y = \{f \in C([a, b]) \,:\, f(a) = 1\}$ a subspace of the linear space $X = C([a, b])$ of continuous complex-valued functions?*

(e) *Is the set $Y = \Pi([a, b])$ of all polynomials $p(x) = a_0 + a_1\,x + a_2\,x^2 + \ldots + a_n\,x^n$, $n \in \mathbb{N}_0$, with complex coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$, a subspace of the space $X = C([a, b])$ of continuous complex-valued functions on $[a, b]$?*

Exercise 5 *Give the proof of Lemma 1.6.*

1.2 Linear Independence and Bases

Finally we revise the notions of linear combination, span, linear independence and linear dependance, and a basis of a linear space and its dimension. While this may initially seem a revision of fairly basic material, we will later-on apply this terminology in the context of (often infinite dimensional) **linear spaces of functions**.

Definition 1.9 (linear combination and span)
*Let X be a linear space over the field $\mathbb{K}$. A **linear combination** of vectors $x_1, x_2, \ldots, x_N$ in X is an expression of the form*

$$\sum_{k=1}^{N} \alpha_k\, x_k = \alpha_1\, x_1 + \alpha_2\, x_2 + \ldots + \alpha_N\, x_N, \qquad \alpha_k \in \mathbb{K}.$$

*The set of all linear combinations of vectors from a set $M \subset X$ is called the **span of** M, denoted by* $\mathrm{span}\,(M)$*.*

We note that $\mathrm{span}\,(M)$ **is a subspace of** X, because it is closed under vector addition and scalar multiplication.

Example 1.10 (linear combination and span)

(a) In $\mathbb{R}^3$, let $M = \{(1,2,3)^T, (2,0,1)^T\}$. Then the span of M is given by

$$\mathrm{span}\, M = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \;:\; \alpha_1, \alpha_2 \in \mathbb{R} \right\}.$$

(b) In the space $C(\mathbb{R})$ of continuous complex-valued functions on $\mathbb{R}$, the polynomial $p(x) = i + 3\,x + (17 + 2\,i)\,x^3$ is a linear combination of the monomials $1, x, x^3$.

(c) The linear space $\Pi_3(\mathbb{R})$ of all polynomials of degree ≤ 3 on $\mathbb{R}$ with complex coefficients is a subspace of $C(\mathbb{R})$ and is the span of the monomials $1, x, x^2, x^3$, that is,

$$\Pi_3(\mathbb{R}) = \mathrm{span}\,\left\{p_0(x) = 1,\, p_1(x) = x,\, p_2(x) = x^2,\, p_3(x) = x^3\right\}.$$

As before $C(\mathbb{R})$ is the space of continuous complex-valued functions on $\mathbb{R}$. □

Definition 1.11 (linear independence and linear dependence)
Let X be a linear space over the field $\mathbb{K}$.

*(i) A subset $M \subset X$ is said to be **linearly independent** if for any finite subset $\{x_1, x_2, \ldots, x_N\} \subset M$ the equality*

$$\sum_{k=1}^{N} \alpha_k\, x_k = \alpha_1\, x_1 + \alpha_2\, x_2 + \ldots + \alpha_N\, x_N = \mathcal{O}$$

is satisfied only for $\alpha_1 = \alpha_2 = \cdots = \alpha_N = 0$.

*(ii) A set $M \subset X$ is said to be **linearly dependent**, if it is not linearly independent.*

Note: If M is linearly dependent, then one of the vectors in M can be written as a linear combination of the others (see Definition 1.11).

Let us consider some examples.

Example 1.12 (linear independence and linear dependence)

(a) The vectors $(1,1,0)^T$, $(3,2,0)^T$, and $(0,0,4)^T$ are linearly independent in $\mathbb{R}^3$.

(b) The vectors $(1,2)^T$, $(1-3)^T$, and $(-1,0)^T$ are linearly dependent in $\mathbb{R}^2$.

(c) In the space $C([a,b])$ of continuous complex-valued functions on $[a,b]$, the polynomials $p_1(x) = x$, $p_2(x) = 1$, $p_3(x) = i + x^2$, and $p_4(x) = x^2$ are linearly dependent, because

$$0\, p_1(x) + (-i)\, p_2(x) + 1\, p_3(x) + (-1)\, p_4(x) = 0 - i + \big(i + x^2\big) - x^2 = 0 \quad \text{for all } x \in [a,b]. \qquad \square$$

Exercise 6 *Investigate which of the following sets are linearly independent in the given linear space. Give a proof of your answer.*

(a) Are the vectors $(0,1)^T$, $(i,0)^T$, and $(1,1)^T$ linearly dependent or linearly independent in $\mathbb{C}^2$?

(b) Are the vectors $(1,2,0)^T$, $(1,-1,-1)^T$, and $(0,0,1)^T$ linearly independent in $\mathbb{R}^3$ or not?

(c) Are the functions $f(x) = x\,\sin x$, $g(x) = \cos x$, $h(x) = \sin x$, and $k(x) = x$ linearly independent in the space $C([a,b])$ of continuous complex-valued functions on $[a,b]$, or not?

Exercise 7 *Show that the set of monomials*

$$M = \big\{1, x, x^2, \ldots, x^n, x^{n+1}, \ldots\big\}$$

is linearly independent in the complex linear space $C(\mathbb{R})$ of continuous complex-valued functions on $\mathbb{R}$.

Definition 1.13 (dimension of a linear space)
Let X be a linear space over the field $\mathbb{K}$.

*(i) The space X is said to be **finite dimensional** if there is a positive integer d such that X contains d linearly independent vectors and every subset M containing more than d vectors is linearly dependent. In this case d is called the **dimension** of X, and we write $d = \dim(X)$.*

*(ii) If X is not finite dimensional, it is **infinite dimensional**.*

Definition 1.14 (basis of a linear space)
*Let X be a finite dimensional linear space over the field $\mathbb{K}$. If a linearly independent subset M of X spans all of X, that is, $\mathrm{span}\,(M) = X$, then M is called a **basis** of X.*

From the last two definitions we can immediately conclude to following corollary.

Corollary 1.15 (characterisation of basis in finite dimensional linear space)
Let X be a finite dimensional linear space with dimension $\dim(X) = d$. Then every set of d linearly independent vectors forms a basis.

Before we give some examples we remark on a consequence of the Definition 1.14 and Corollary 1.15: If X is a d-dimensional space, and if $\{e_1, e_2, \ldots, e_d\}$ is a basis of X, then every $x \in X$ can be **uniquely** represented as

$$x = \sum_{k=1}^{d} \alpha_k \, e_k,$$

with **uniquely determined** coefficients $\alpha_1, \alpha_2, \ldots, \alpha_d \in \mathbb{K}$.

Example 1.16 (canonical basis in $\mathbb{R}^d$)
Let $\mathbb{R}^d$ be the d-dimensional Euclidean space. Then the vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0, 0)^T, \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0, 0)^T, \quad \ldots, \quad \mathbf{e}_d = (0, 0, 0, \ldots, 0, 1)^T,$$

(where $\mathbf{e}_j$ has the jth entry 1 and all other entries zero) form the **canonical basis** of $\mathbb{R}^d$. □

Example 1.17 (infinite dimensional function spaces)
The space $C([a, b])$ of continuous complex-valued functions on $[a, b]$ is infinite dimensional. □

Exercise 8 *Let* $\Pi(\mathbb{R}) = \text{span}\,\{1, x, x^2, x^3, \ldots, x^n, \ldots\} \subset C(\mathbb{R})$ *denote the set of all polynomials on* $\mathbb{R}$ *with complex coefficients, and let*

$$\Pi_n(\mathbb{R}) = \text{span}\,\{1, x, x^2, x^3, \ldots, x^n\}$$

be the subset of those polynomials on $\mathbb{R}$ *of degree* $\leq n$ *with complex coefficients.*

(a) Show that $\Pi_n(\mathbb{R})$ *is a finite dimensional complex linear space and find its dimension.*

(b) Show that $\Pi(\mathbb{R})$ *is an infinite dimensional complex linear space.*

Exercise 9 *Proof Corollary 1.15.*

Chapter 2

Normed Linear Spaces and Their Topology

In this chapter we review some important concepts that you will most likely have already encountered previously in other courses. Section 2.1 reviews material on **norms** and **normed linear spaces**. Here we introduce the p-norms for the sequence spaces $\ell_p(\mathbb{N})$ and the $L_p([a,b])$-norms and $L_p(\mathbb{R})$-norms for the spaces $L_p([a,b])$ and $L_p(\mathbb{R})$ of functions whose pth powers are Lebesgue integrable over $[a,b]$ and $\mathbb{R}$, respectively. In Section 2.2 we encounter important inequalities, namely **Hölder's inequality** and the **Minkowski inequality**, that will be used throughout the course and that allow us to show that the $\ell_p(\mathbb{N})$, $L_p([a,b])$, and $L_p(\mathbb{R})$ are indeed normed linear spaces. In Section 2.3 we revise material on **elementary topology** about open and closed sets, accumulation points, the closure of a set, and dense sets and separable sets in a linear space. In Section 2.4 we finally discuss the notions of **convergence** and **completeness** in a normed linear space. All concepts will be illustrated with examples.

As in the previous chapter the field $\mathbb{K}$ is either $\mathbb{R}$ in which case we consider a real linear space or $\mathbb{C}$ in which case we consider a complex linear space.

2.1 Norms

Definition 2.1 (norm and normed linear space)
A ***norm*** *on a linear space X over the field $\mathbb{K}$ is a real-valued function $\|\cdot\| : X \to \mathbb{R}$, satisfying the following conditions:*

(i) $\|x\| \geq 0$ *for all* $x \in X$.

(ii) $\|x\| = 0$ *if and only if* $x = \mathcal{O}$ (***non-degeneracy***).

(iii) $\|\alpha\, x\| = |\alpha|\, \|x\|$ *for all* $\alpha \in \mathbb{K}$ *and all* $x \in X$.

(iv) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$ (***triangle inequality***).

The vector space X together with a norm $\|\cdot\| : X \to \mathbb{R}$ is called a ***normed linear space*** *(or* ***normed vector space****).*

Note that the triangle inequality implies the **lower triangle inequality**

$$\big|\|x\| - \|y\|\big| \leq \|x - y\| \qquad \text{for all } x, y \in X.$$

Example 2.2 (Euclidean norm on $\mathbb{R}^d$)
The **Euclidean norm** on $\mathbb{R}^d$ is defined by

$$\|\mathbf{x}\|_2 = \left(\sum_{k=1}^{d} |x_k|^2\right)^{1/2},$$

which is the (geometric) length of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_d)^T$. □

Example 2.3 (other norms for $\mathbb{R}^d$ and $\mathbb{C}^d$)
The linear space $\mathbb{R}^d$ (or $\mathbb{C}^d$) is a normed vector space with each of the following norms

$$\|\mathbf{x}\|_p := \left(\sum_{k=1}^{d} |x_k|^p\right)^{1/p}, \qquad 1 \leq p < \infty, \tag{2.1.1}$$

and

$$\|\mathbf{x}\|_\infty := \max_{k=1,2,\ldots,d} |x_k|. \tag{2.1.2}$$

For $p = 2$, we have the special case of the Euclidean norm. Apart from $p = 1$ and $p = \infty$ it is not trivial to verify that these functions are actually norms for $\mathbb{R}^d$ (or $\mathbb{C}^d$). The difficult property is the triangle inequality, and we will learn in the next section how to verify it. □

Exercise 10 *Verify that $\mathbb{C}^d$ with the norm $\|\cdot\|_1$, defined by (2.1.1) with $p = 1$, is a complex normed linear space.*

Exercise 11 *Verify that $\mathbb{C}^d$ with the norm $\|\cdot\|_\infty$, defined by (2.1.2), is a complex normed linear space.*

Definition 2.4 (equivalent norms)
Let X be a linear space and let $\|\cdot\| : X \to \mathbb{K}$ and $|||\cdot||| : X \to \mathbb{K}$ denote two different norms for X. The norms $\|\cdot\|$ and $|||\cdot|||$ for X are called ***equivalent*** *(or* ***equivalent norms****), if there exist two positive real constants c_1 and c_2 such that*

$$c_1 \|x\| \leq |||x||| \leq c_2 \|x\| \qquad \text{for all } x \in X.$$

It is important to be aware of the following lemma which will not be proved in this course.

Lemma 2.5 (all norms on a finite dimensional linear space are equivalent)
On any ***finite dimensional*** *linear space all norms are equivalent.*

It should be noted that the above lemma does **not** hold for infinite dimensional spaces!

We are going to introduce **spaces of sequences** $x = (x_1, x_2, \ldots, x_n, \ldots) = (x_k)_{k\in\mathbb{N}}$, where $x_k \in \mathbb{K}$ for all $k \in \mathbb{N}$ (see also Example 1.3).

Definition 2.6 (sequence spaces $\ell_p(\mathbb{N})$, where $1 \le p \le \infty$)
*Let $1 \le p < \infty$. A sequence $x = (x_1, x_2, x_3, \ldots) = (x_k)_{k\in\mathbb{N}}$, where $x_k \in \mathbb{K}$ for all $k \in \mathbb{N}$, belongs to the **sequence space** $\ell_p(\mathbb{N})$ if*

$$\|x\|_p := \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{1/p} \tag{2.1.3}$$

*is finite. A sequence $x = (x_1, x_2, x_3, \ldots) = (x_k)_{k\in\mathbb{N}}$,where $x_k \in \mathbb{K}$ for all $k \in \mathbb{N}$, belongs to the **sequence space** $\ell_\infty(\mathbb{N})$ if*

$$\|x\|_\infty := \sup_{k\in\mathbb{N}} |x_k| \tag{2.1.4}$$

*is finite. The functions $\|\cdot\|_p$ in (2.1.3) for $1 \le p < \infty$ and $\|\cdot\|_\infty$ in (2.1.4) for $p = \infty$, respectively, are called the $\ell_p(\mathbb{N})$-**norms**.*

However, it should be noted that we have not yet verified that $\|\cdot\|_p$ is actually a norm for $\ell_p(\mathbb{N})$. This is not obvious and is only straightforward to verify for $\ell_1(\mathbb{N})$ and $\ell_\infty(\mathbb{N})$. To show that $\|\cdot\|_p$ is a norm for $\ell_p(\mathbb{N})$ with $1 < p < \infty$, we need the inequalities that are proved in the next section.

Example 2.7 (sequence spaces $\ell_1(\mathbb{N})$ and $\ell_2(\mathbb{N})$)
The sequence $x = (1/k)_{k\in\mathbb{N}}$ is in $\ell_2(\mathbb{N})$ but not in $\ell_1(\mathbb{N})$, because

$$\|x\|_2 = \sum_{k=0}^{\infty} \left|\frac{1}{k}\right|^2 = \sum_{k=0}^{\infty} \frac{1}{k^2} < \infty,$$

whereas

$$\|x\|_1 = \sum_{k=0}^{\infty} \left|\frac{1}{k}\right| = \sum_{k=0}^{\infty} \frac{1}{k} = \infty. \qquad \square$$

Exercise 12 *Show that $\ell_1(\mathbb{N})$ with $\|\cdot\|_1$, defined by (2.1.3) with $p = 1$, is a normed linear space.*

Exercise 13 *Show that $\ell_\infty(\mathbb{N})$ with $\|\cdot\|_\infty$, defined by (2.1.4), is a normed linear space.*

Important examples of normed linear function spaces are $C([a, b])$ with the supremum norm and $L_p([a, b])$ defined below.

Example 2.8 (space $C([a, b])$ with the supremum norm)
The space $C([a, b])$ of continuous complex-valued functions on $[a, b]$ with the **supremum norm**

$$\|f\|_{C[a,b]} := \max_{x\in[a,b]} |f(x)|, \qquad f \in C([a, b]), \tag{2.1.5}$$

is a normed linear space. $\square$

Exercise 14 *Verify that the space $C([a,b])$ of continuous complex-valued functions on $[a,b]$ with the supremum norm (2.1.5) is a normed linear space.*

Note: The theorem below mentions the so-called **essential supremum** in

$$\operatorname*{ess-sup}_{x\in[a,b]} |f(x)|.$$

When taking the essential supremum of $|f(x)|$ on $[a,b]$, we may ignore the values of f on sets of Lebesgue measure zero. For example, finite sets of points in $\mathbb{R}$ or the set $\mathbb{N}$ have Lebesgue measure zero. To get a full understanding of sets of Lebesgue measure zero you will need to learn about the Lebesgue integral. However, we will not use the spaces $L_\infty([a,b])$ and $L_\infty(\mathbb{R})$ a lot (if at all) in this course; so not having covered the Lebesgue integral in other courses will not cause any problems.

Definition 2.9 (spaces $L_p([a,b])$, where $1 \le p \le \infty$)
Let $1 \le p < \infty$. The space of those measurable complex-valued functions defined on the interval $[a,b]$ for which

$$\|f\|_{L_p([a,b])} := \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p} \tag{2.1.6}$$

is finite, is denoted by $L_p([a,b])$. If $p=\infty$, we define

$$\|f\|_{L_\infty([a,b])} := \operatorname*{ess-sup}_{x\in[a,b]} |f(x)|, \tag{2.1.7}$$

*and denote the space of those complex-valued functions defined on the interval $[a,b]$ for which $\|f\|_{L_\infty([a,b])} < \infty$ by $L_\infty([a,b])$. The functions $\|\cdot\|_{L_p([a,b])}$ defined by (2.1.6) for $1 \le p < \infty$ and by (2.1.7) for $p=\infty$ are called the $L_p([a,b])$-**norms**.*

Analogously we define L_p-spaces on $\mathbb{R}$.

Definition 2.10 (spaces $L_p(\mathbb{R})$, where $1 \le p \le \infty$)
Let $1 \le p < \infty$. The space of those measurable complex-valued functions defined on $\mathbb{R}$ for which

$$\|f\|_{L_p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x\right)^{1/p} \tag{2.1.8}$$

is finite, is denoted by $L_p(\mathbb{R})$. If $p=\infty$, we define

$$\|f\|_{L_\infty(\mathbb{R})} := \operatorname*{ess-sup}_{x\in\mathbb{R}} |f(x)|, \tag{2.1.9}$$

*and denote the space of those complex-valued functions defined on $\mathbb{R}$ for which $\|f\|_{L_\infty(\mathbb{R})} < \infty$ by $L_\infty(\mathbb{R})$. The functions $\|\cdot\|_{L_p(\mathbb{R})}$ defined by (2.1.8) for $1 \le p < \infty$ and by (2.1.9) for $p=\infty$ are called the $L_p(\mathbb{R})$-**norms**.*

As for the spaces $\ell_p(\mathbb{N})$, we need again to be careful, as only for $p=1$ and $p=\infty$ it is easily verified that $L_p([a,b])$ and $L_p(\mathbb{R})$ are normed linear spaces. For $1 < p < \infty$, we will see in the next section how to verify that $\|\cdot\|_{L_p}$ is indeed a norm.

Example 2.11 ($L_1([0,1])$ and $L_2([0,1])$)
The function $f(x) = 1/\sqrt{x} = x^{-1/2}$ is in $L_1([0,1])$ but not in $L_2([0,1])$. Before we verify this we note that the fact that the function has a singularity at $x = 0$ is in itself not a problem, since $\{0\}$ is a set of Lebesgue measure zero. (Even for the Riemann integral, the value of a function at a single individual point can be ignored.) We compute the norms

$$\|f\|_{L_1([0,1])} = \int_0^1 |x^{-1/2}|\,\mathrm{d}x = \int_0^1 x^{-1/2}\,\mathrm{d}x = \left[2\,x^{1/2}\right]_0^1 = 2 < \infty$$

and

$$\|f\|_{L_2([0,1])} = \int_0^1 |x^{-1/2}|^2\,\mathrm{d}x = \int_0^1 x^{-1}\,\mathrm{d}x = [\ln(x)]_0^1 = \infty.$$

Hence clearly, $f \in L_1([0,1])$ and $f \notin L_2([0,1])$. □

Remark 2.12 (L_p-spaces for more general sets)
It is clear that, for $1 \le p < \infty$ and 'more general sets' $\Omega \subset \mathbb{R}$, we can also define the space $L_p(\Omega)$ as the set of all those measurable functions on Ω for which

$$\|f\|_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p\,\mathrm{d}x\right)^{1/p} \tag{2.1.10}$$

is finite. It can then be verified (analogously to the cases $\Omega = [a,b]$ and $\Omega = \mathbb{R}$) that (2.1.10) is a norm and that hence $L_p(\Omega)$ is a normed linear space. By 'more general sets' we mean 'Lebesgue measurable sets'; these include in particular all open and half-open intervals.

Exercise 15 *Let the sequence $x = (x_k)_{k\in\mathbb{N}}$ be defined by $x_k = k^{-\beta}$, $k \in \mathbb{N}$, with a real number $\beta > 0$.*

(a) Find the values of β for which $x \in \ell_1(\mathbb{N})$.

(b) Find the values of β for which $x \in \ell_p(\mathbb{N})$ for a given real p with $1 < p < \infty$.

Exercise 16 *Let $f(x) := x^{-\alpha}$, $x \in (0,\infty)$, with a real number $\alpha > 0$.*

(a) Find the values of α for which $f \in L\big((0,1)\big)$.

(b) Find the values of α for which $f \in L\big((1,\infty)\big)$.

Exercise 17 *Check whether the function*

$$f(\mathbf{x}) = \sqrt{|x_1|^2 + |x_2|^2} + \sqrt{|x_1\,x_2|}, \qquad \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2,$$

defines a norm for $\mathbb{R}^2$. Give a proof of your answer.

Exercise 18 *Show that the space $L_1([a,b])$ the norm $\|\cdot\|_{L_1([a,b])}$ defined by (2.1.6) with $p = 1$, is a normed linear space. You do not have to show the non-degeneracy of the norm, as this requires knowledge of the Lebesgue integral.*

Exercise 19 *Consider the linear space $L_1([0,1])$. Is the supremum norm*

$$\|f\|_{C([0,1])} := \sup_{x\in[0,1]} |f(x)|$$

a norm for $L_1([0,1])$? Give a proof of your answers! (If you do not know about the Lebesgue integral, then considering the Riemann integral will be sufficient to answer this question.)

2.2 Hölder's and Minkowski's Inequalities and the Spaces $\ell_p(\mathbb{N})$, $L_p([a,b])$ and $L_p(\mathbb{R})$

To be able to prove that $\ell_p(\mathbb{N})$, $L_p([a,b])$, and $L_p(\mathbb{R})$, where $1 \le p < \infty$, are **normed linear spaces**, we need to prove **Hölder's inequality** and the **Minkowski inequality**. These will allow us to verify the triangle inequality in the spaces $\ell_p(\mathbb{N})$, $L_p([a,b])$, and $L_p(\mathbb{R})$.

In this section we use the following notation: $1 \le p,q \le \infty$ are real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where the convention is that if $p = 1$ then $q = \infty$ and vice versa. Then the numbers p and q are called **conjugate exponents**. Note that then

$$q - 1 = \frac{1}{p-1} \qquad \text{and} \qquad (p-1)\,q = p.$$

Lemma 2.13 (Young's inequality)
Let $1 < p,q < \infty$ be conjugate exponents, that is, $1/p + 1/q = 1$. Then for any non-negative real numbers a and b we have

$$a\,b \le \frac{a^p}{p} + \frac{b^q}{q}.$$

Young's inequality will help us to prove the important Hölder's inequality.

Proof of Lemma 2.13: Recall that the exponential function $f(t) = \exp(t)$ is convex, that is, for any $\lambda \in [0,1]$ and any $s,t \in \mathbb{R}$, we have

$$f(\lambda s + (1-\lambda)t) \le \lambda f(s) + (1-\lambda) f(t). \qquad (2.2.1)$$

Using this and $\ln(x\,y) = \ln(x) + \ln(y)$, $\ln(x^y) = y\,\ln(x)$, and $1/p + 1/q = 1$, we find

$$\begin{aligned} a\,b &= \exp\big(\ln(a\,b)\big) = \exp\big(\ln(a) + \ln(b)\big) = \exp\left(\frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q)\right) \\ &\le \frac{1}{p}\exp\big(\ln(a^p)\big) + \frac{1}{q}\exp\big(\ln(b^q)\big) = \frac{1}{p}\,a^p + \frac{1}{q}\,b^q, \end{aligned}$$

where we have used (2.2.1) with $\lambda = 1/p$ and $1-\lambda = 1-1/p = 1/q$ in the second last step. □

Now we can prove Hölder's inequality with the help of Young's inequality.

Lemma 2.14 (Hölder's inequality for $\mathbb{R}^d$ and $\mathbb{C}^d$ and the sequence spaces $\ell_p(\mathbb{N})$)
Let $1 \le p, q \le \infty$ be conjugate exponents, that is, $1/p + 1/q = 1$.

(i) Then for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ (or $\mathbf{x}, \mathbf{y} \in \mathbb{C}^d$)

$$\sum_{k=1}^{d} |x_k\, y_k| \le \left(\sum_{k=1}^{d} |x_k|^p \right)^{1/p} \left(\sum_{k=1}^{d} |y_k|^q \right)^{1/q} = \|\mathbf{x}\|_p \, \|\mathbf{y}\|_q, \qquad 1 < p, q < \infty, \tag{2.2.2}$$

and for $p = 1$ and $q = \infty$

$$\sum_{k=1}^{d} |x_k\, y_k| \le \left(\sum_{k=1}^{d} |x_k| \right) \left(\sup_{k=1,2,\ldots,d} |y_k| \right) = \|\mathbf{x}\|_1 \, \|\mathbf{y}\|_\infty. \tag{2.2.3}$$

(ii) Then for any sequences $x = (x_k)_{k\in\mathbb{N}} \in \ell_p(\mathbb{N})$ and $y = (y_k)_{k\in\mathbb{N}} \in \ell_q(\mathbb{N})$ we have

$$\sum_{k\in\mathbb{N}} |x_k\, y_k| \le \left(\sum_{k\in\mathbb{N}} |x_k|^p \right)^{1/p} \left(\sum_{k\in\mathbb{N}} |y_k|^q \right)^{1/q} = \|x\|_p \, \|y\|_q, \qquad 1 < p, q < \infty, \tag{2.2.4}$$

and for $p = 1$ and $q = \infty$

$$\sum_{k\in\mathbb{N}} |x_k y_k| \le \left(\sum_{k\in\mathbb{N}} |x_k| \right) \left(\sup_{k\in\mathbb{N}} |y_k| \right) = \|x\|_1 \, \|y\|_\infty. \tag{2.2.5}$$

The estimates (2.2.2) and (2.2.3), (2.2.4) and (2.2.5) are called ***Hölder's inequality****. In the special case $p = q = 2$, Hölder's inequality is the* ***Cauchy-Schwarz inequality****.*

We have an analogous lemma for the function spaces $L_p([a,b])$ and $L_p(\mathbb{R})$.

Lemma 2.15 (Hölder's inequality for $L_p([a,b])$)
Let $1 \le p, q \le \infty$ be conjugate exponents, that is, $1/p + 1/q = 1$. For any functions $f \in L_p([a,b])$ and $g \in L_q([a,b])$ we have for $1 < p, q < \infty$

$$\int_a^b |f(x)\, g(x)| \,\mathrm{d}x \le \left(\int_a^b |f(x)|^p \,\mathrm{d}x \right)^{1/p} \left(\int_a^b |g(x)|^q \,\mathrm{d}x \right)^{1/q} = \|f\|_{L_p([a,b])} \, \|g\|_{L_q([a,b])}, \tag{2.2.6}$$

and for $p = 1$ and $q = \infty$

$$\int_a^b |f(x)\, g(x)| \,\mathrm{d}x \le \left(\int_a^b |f(x)| \,\mathrm{d}x \right) \left(\operatorname*{ess-sup}_{x\in[a,b]} |g(x)| \right) = \|f\|_{L_1([a,b])} \, \|g\|_{L_\infty([a,b])}. \tag{2.2.7}$$

The estimates (2.2.6) and (2.2.7) are called ***Hölder's inequality****. In the special case $p = q = 2$, Hölder's inequality is the* ***Cauchy-Schwarz inequality****.*

Lemma 2.16 (Hölder's inequality for $L_p(\mathbb{R})$)
Let $1 \le p,q \le \infty$ be conjugate exponents, that is, $1/p + 1/q = 1$. For any functions $f \in L_p(\mathbb{R})$ and $g \in L_q(\mathbb{R})$ we have for $1 < p,q < \infty$

$$\int_{\mathbb{R}} |f(x)\, g(x)|\, \mathrm{d}x \le \left(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x \right)^{1/p} \left(\int_{\mathbb{R}} |g(x)|^q \, \mathrm{d}x \right)^{1/q} = \|f\|_{L_p(\mathbb{R})} \, \|g\|_{L_q(\mathbb{R})}, \qquad (2.2.8)$$

and for $p = 1$ and $q = \infty$

$$\int_{\mathbb{R}} |f(x)\, g(x)|\, \mathrm{d}x \le \left(\int_{\mathbb{R}} |f(x)| \, \mathrm{d}x \right) \left(\operatorname{ess-sup}_{x \in \mathbb{R}} |g(x)| \right) = \|f\|_{L_1(\mathbb{R})} \, \|g\|_{L_\infty(\mathbb{R})}. \qquad (2.2.9)$$

*The estimates (2.2.8) and (2.2.9) are called **Hölder's inequality**. In the special case $p = q = 2$, Hölder's inequality is the **Cauchy-Schwarz inequality**.*

Proof of Lemma 2.14: We proof the result only in the case of sequences. By setting $x_k = 0$ for $k > d$, (2.2.4) and (2.2.5) then immediately imply (2.2.2) and (2.2.3), respectively.

We start with the observation that if $x = (0)_{k \in \mathbb{N}} = (0,0,\ldots)$ or if $y = (0)_{k\in\mathbb{N}} = (0,0,\ldots)$, respectively, then in (2.2.4) and (2.2.5) the left hand-side is zero. Also we have $\|x\|_p = 0$ or $\|y\|_q = 0$, respectively, and hence the right-hand side is zero. Thus if $x = (0)_{k\in\mathbb{N}}$ or $y = (0)_{k\in\mathbb{N}}$, then Hölder's inequality becomes an equality and is trivially true.

Now assume that $x = (x_k)_{k\in\mathbb{N}}$ and $y = (y_k)_{k\in\mathbb{N}}$ are both different from the zero sequence.

For the special case $p = 1$ and $q = \infty$ we have for each $k \in \mathbb{N}$ the estimate

$$|x_k\, y_k| \le |x_k|\, |y_k| \le |x_k| \sup_{m\in\mathbb{N}} |y_m| = |x_k|\, \|y\|_\infty,$$

since $y = (y_k)_{k\in\mathbb{N}} \in \ell_\infty(\mathbb{N})$. Since $x = (x_k)_{k\in\mathbb{N}} \in \ell_1(\mathbb{N})$, we know that $(|x_k|\, \|y\|_\infty)_{k\in\mathbb{N}}$ is also in $\ell_1(\mathbb{N})$. Thus from the dominated converge theorem for series,

$$\sum_{k\in\mathbb{N}} |x_k\, y_k| \le \sum_{k\in\mathbb{N}} |x_k|\, |y_k| \le \sum_{k\in\mathbb{N}} |x_k|\, \|y\|_\infty = \|y\|_\infty \sum_{k\in\mathbb{N}} |x_k| = \|x\|_1\, \|y\|_\infty,$$

which verifies (2.2.5).

For $1 < p,q < \infty$, it is not clear that the sum on the left-hand side of Hölder's inequality converges; thus we consider its partial sums: using Young's inequality we derive

$$\begin{aligned}
\frac{1}{\|x\|_p\, \|y\|_q} \sum_{k=1}^n |x_k\, y_k| &= \sum_{k=1}^n \frac{|x_k|}{\|x\|_p} \, \frac{|y_k|}{\|y\|_q} \\
&\le \sum_{k=1}^n \left(\frac{1}{p} \, \frac{|x_k|^p}{\|x\|_p^p} + \frac{1}{q}\, \frac{|y_k|^q}{\|y\|_q^q} \right) \\
&= \sum_{k=1}^n \frac{1}{p}\, \frac{|x_k|^p}{\|x\|_p^p} + \sum_{k=1}^n \frac{1}{q}\, \frac{|y_k|^q}{\|y\|_q^q}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{p}\,\frac{1}{\|x\|_p^p}\sum_{k=1}^{n}|x_k|^p + \frac{1}{q}\,\frac{1}{\|y\|_q^q}\sum_{k=1}^{n}|y_k|^q \\
&\le \frac{1}{p}\,\frac{1}{\|x\|_p^p}\sum_{k=1}^{\infty}|x_k|^p + \frac{1}{q}\,\frac{1}{\|y\|_q^q}\sum_{k=1}^{\infty}|y_k|^q \\
&= \frac{1}{p}\,\frac{1}{\|x\|_p^p}\,\|x\|_p^p + \frac{1}{q}\,\frac{1}{\|y\|_q^q}\,\|y\|_q^q \\
&= \frac{1}{p} + \frac{1}{q} \;=\; 1.
\end{aligned}$$

Multiplying this equation with $\|x\|_p\,\|y\|_q$ gives

$$\sum_{k=1}^{n}|x_k\,y_k| \le \|x\|_p\,\|y\|_q \qquad \text{for all } n\in\mathbb{N}. \tag{2.2.10}$$

As the partial sums $s_n := \sum_{k=1}^{n}|x_k|\,|y_k|$ form an increasing sequence $(s_n)_{n\in\mathbb{N}}$ of real numbers which, from (2.2.10), is bounded from above, we know that the series converges and that the upper bound is also valid for the limit. Hence letting $k\to\infty$ in (2.2.10) gives

$$\sum_{k=1}^{\infty}|x_k\,y_k| \le \|x\|_p\,\|y\|_q$$

which proves the stated result. □

Lemma 2.15 and Lemma 2.16 are proved analogously.

With the help of Hölder's inequality we can prove the Minkowski inequality which provides the triangle inequality for the norms $\|\cdot\|_p$, $\|\cdot\|_{L_p([a,b])}$, and $\|\cdot\|_{L_p(\mathbb{R})}$.

Lemma 2.17 (Minkowski Inequality for $\mathbb{R}^n$, $\mathbb{C}^n$ and $\ell_p(\mathbb{N})$)
Let $1\le p<\infty$.

(i) For any vectors $\mathbf{x},\mathbf{y}\in\mathbb{R}^d$ (or $\mathbf{x},\mathbf{y}\in\mathbb{C}^d$), we have

$$\left(\sum_{k=1}^{d}|x_k+y_k|^p\right)^{1/p} \le \left(\sum_{k=1}^{d}|x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{d}|y_k|^p\right)^{1/p}. \tag{2.2.11}$$

(ii) For any sequences $x=(x_k)_{k\in\mathbb{N}}$ and $y=(y_k)_{k\in\mathbb{N}}$ in $\ell_p(\mathbb{N})$, we have

$$\left(\sum_{k\in\mathbb{N}}|x_k+y_k|^p\right)^{1/p} \le \left(\sum_{k\in\mathbb{N}}|x_k|^p\right)^{1/p} + \left(\sum_{k\in\mathbb{N}}|y_k|^p\right)^{1/p}. \tag{2.2.12}$$

Analogously we have for the spaces $L_p([a,b])$ and $L_p(\mathbb{R})$ a Minkowski's inequality.

Lemma 2.18 (Minkowski Inequality for $L_p([a,b])$ and $L_p(\mathbb{R})$)
Let $1 \leq p < \infty$.

(i) For any functions $f, g \in L_p([a,b])$ we have

$$\left(\int_a^b |f(x)+g(x)|^p \, \mathrm{d}x\right)^{1/p} \leq \left(\int_a^b |f(x)|^p \, \mathrm{d}x\right)^{1/p} + \left(\int_a^b |g(x)|^p \, \mathrm{d}x\right)^{1/p}. \tag{2.2.13}$$

(ii) For any functions $f, g \in L_p(\mathbb{R})$ we have

$$\left(\int_{\mathbb{R}} |f(x)+g(x)|^p \, \mathrm{d}x\right)^{1/p} \leq \left(\int_{\mathbb{R}} |f(x)|^p \, \mathrm{d}x\right)^{1/p} + \left(\int_{\mathbb{R}} |g(x)|^p \, \mathrm{d}x\right)^{1/p}. \tag{2.2.14}$$

Proof of Lemma 2.17: Again, we prove the result only in the case of sequences. By setting $x_k = 0$ for $k > d$, (2.2.12) then immediately implies (2.2.11).

For $p = 1$, the Minkowski inequality follows straightforward from the triangle inequality for real numbers: As $|x_x + y_k| \leq |x_k| + |y_k|$ we have

$$\sum_{k\in\mathbb{N}} |x_k + y_k| \leq \sum_{k\in\mathbb{N}} \big(|x_k| + |y_k|\big) = \sum_{k\in\mathbb{N}} |x_k| + \sum_{k\in\mathbb{N}} |y_k|.$$

Now let $1 < p < \infty$. First we observe that for $x + y = (0)_{k\in\mathbb{N}}$ the estimate (2.2.12) is trivially true. Thus we assume from now on that $x + y \neq (0)_{k\in\mathbb{N}}$. As we do not yet know that the sequence $x + y = (x_k + y_k)_{k\in\mathbb{N}}$ is in $\ell_p(\mathbb{N})$, we start by considering the partial sums

$$s_n := \sum_{k=1}^{n} |x_k + y_k|^p$$

We choose q such that p and q are conjugate, that is, $1/p + 1/q = 1$. From the triangle inequality for complex numbers, we find

$$\begin{aligned} \sum_{k=1}^{n} |x_k + y_k|^p &= \sum_{k=0}^{n} |x_k + y_k|^{p-1}\, |x_k + y_k| \\ &\leq \sum_{k=0}^{n} |x_k + y_k|^{p-1} \big(|x_k| + |y_k|\big) \\ &= \sum_{k=0}^{n} |x_k + y_k|^{p-1}\, |x_k| + \sum_{k=0}^{n} |x_k + y_k|^{p-1}\, |y_k|. \end{aligned} \tag{2.2.15}$$

Since the sum in (2.2.15) is finite, we can use Hölder's inequality (note $(p-1)q = p$)

$$\begin{aligned} \sum_{k=1}^{n} |x_k + y_k|^p &= \sum_{k=0}^{n} |x_k + y_k|^{p-1}\, |x_k| + \sum_{k=0}^{n} |x_k + y_k|^{p-1}\, |y_k| \\ &\leq \left(\sum_{k=0}^{n} |x_k|^p\right)^{1/p} \left(\sum_{k=0}^{n} |x_k + y_k|^{(p-1)q}\right)^{1/q} + \left(\sum_{k=0}^{n} |y_k|^p\right)^{1/p} \left(\sum_{k=0}^{n} |x_k + y_k|^{(p-1)q}\right)^{1/q} \end{aligned}$$

$$\leq \left(\sum_{k=0}^{\infty}|x_k|^p\right)^{1/p}\left(\sum_{k=0}^{n}|x_k+y_k|^p\right)^{1/q}+\left(\sum_{k=0}^{\infty}|y_k|^p\right)^{1/p}\left(\sum_{k=0}^{n}|x_k+y_k|^p\right)^{1/q}$$
$$= \Big(\|x\|_p+\|y\|_p\Big)\left(\sum_{k=0}^{n}|x_k+y_k|^p\right)^{1/q}.$$

We know that the upper bound is finite, since $x = (x_k)_{k\in\mathbb{N}}$ and $y = (y_k)_{k\in\mathbb{N}}$ are in $\ell_p(\mathbb{N})$. Dividing by the second factor in the last line (which non-zero for large enough n) and using $1 - 1/q = 1/p$, gives

$$\left(\sum_{k=0}^{n}|x_k+y_k|^p\right)^{1/p} = \left(\sum_{k=0}^{n}|x_k+y_k|^p\right)^{1-1/q} \leq \|x\|_p+\|y\|_p \qquad \text{for all } n \geq N,$$

where N is the smallest positive integer for which that $x_N + y_N \neq 0$. Since this estimate estimate is uniformly in n and since the left-hand side increases with $n \geq N$, we may take the limit for $n \to \infty$ on the left-hand side, and the estimate is still satisfied in the limit. Thus

$$\|x+y\|_p = \left(\sum_{k=0}^{\infty}|x_k+y_k|^p\right)^{1/p} \leq \|x\|_p+\|y\|_p$$

which proves desired result. □

Lemma 2.2.13 is proved analogously.

The lemmas above provide us with tools to verify that the spaces $\mathbb{R}^d$ (or $\mathbb{C}^n$) with the p-norm $\|\cdot\|_p$, and the sequence spaces $\ell_p(\mathbb{N})$ and the function spaces $L_p([a,b])$ and $L_p(\mathbb{R})$ are normed linear spaces.

Theorem 2.19 ($\mathbb{R}^d$ and $\mathbb{C}^d$ with the p-norm $\|\cdot\|_p$ are normed linear spaces)
Let $1 \leq p < \infty$. The space $\mathbb{R}^d$ (or $\mathbb{C}^d$) with the function $\|\cdot\|_p : \mathbb{R}^d \to \mathbb{R}$ (or $\|\cdot\|_p : \mathbb{C}^d \to \mathbb{R}$),

$$\|\mathbf{x}\|_p := \left(\sum_{k=1}^{d}|x_k|^p\right)^{1/p}, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_d)^T,$$

is a ***normed linear space****.*
The space $\mathbb{R}^d$ (or $\mathbb{C}^d$) with the function $\|\cdot\|_\infty : \mathbb{R}^d \to \mathbb{R}$ (or $\|\cdot\|_\infty : \mathbb{C}^d \to \mathbb{R}$),

$$\|\mathbf{x}\|_\infty := \sup_{k=1,2,\ldots,d}|x_k|, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_d)^T,$$

is a ***normed linear space****.*

Theorem 2.20 ($\ell_p(\mathbb{N})$ with the p-norm $\|\cdot\|_p$ is a normed linear space)
Let $1 \le p < \infty$. The sequence space $\ell_p(\mathbb{N})$ with the function $\|\cdot\|_p : \ell_p(\mathbb{N}) \to \mathbb{R}$,

$$\|x\|_p := \left(\sum_{k\in\mathbb{N}} |x_k|^p \right)^{1/p}, \qquad x = (x_k)_{k\in\mathbb{N}} \in \ell_p(\mathbb{N}), \tag{2.2.16}$$

*is a **normed linear space**.*
The sequence space $\ell_\infty(\mathbb{N})$ with the function $\|\cdot\|_\infty : \ell_\infty(\mathbb{N}) \to \mathbb{R}$,

$$\|x\|_\infty := \sup_{k\in\mathbb{N}} |x_k|, \qquad x = (x_k)_{k\in\mathbb{N}} \in \ell_\infty(\mathbb{N}), \tag{2.2.17}$$

*is a **normed linear space**.*

We only prove Theorem 2.20, as the proof of Theorem 2.19 is completely analogous.

Proof of Theorem 2.20: We only discuss the case $1 < p < \infty$, as the cases $p = 1$ and $p = \infty$ were discussed as exercises in the previous section.

First we have to give some thought to the fact why $\ell_p(\mathbb{N})$ is a vector space. Since we know (see Exercise 1.3)) that the space $\ell(\mathbb{N})$ of all sequence $x = (x_k)_{k\in\mathbb{N}}$ in $\mathbb{K}$ is a linear space and since $\ell_p(\mathbb{N}) \subset \ell(\mathbb{N})$, to verify that $\ell_p(\mathbb{N})$ is a linear space, we only have to show that $x + y \in \ell_p(\mathbb{N})$ and $\alpha\, x \in \ell_p(\mathbb{N})$ for all $x, y \in \ell_p(\mathbb{N})$ and $\alpha \in \mathbb{K}$.

As $\alpha\,(x_k)_{k\in\mathbb{N}} = (\alpha\, x_k)_{k\in\mathbb{N}}$, we have

$$\|\alpha\, x\|_p = \left(\sum_{k\in\mathbb{N}} |\alpha\, x_k|^p \right)^{1/p} = \left(|\alpha|^p \sum_{k\in\mathbb{N}} |x_k|^p \right)^{1/p} = |\alpha| \left(\sum_{k\in\mathbb{N}} |x_k|^p \right)^{1/p} = |\alpha|\, \|x\|_p. \tag{2.2.18}$$

Thus for $x = (x_k)_{k\in\mathbb{N}} \in \ell_p(\mathbb{N})$ and $\alpha \in \mathbb{K}$, we have $\|\alpha\, x\|_p = |\alpha|\, \|x\|_p < \infty$ and hence $\alpha\, x \in \ell_p(\mathbb{N})$.

The Minkowski inequality ensures that for any $x = (x_k)_{k\in\mathbb{N}}$ and $y = (y_k)_{k\in\mathbb{N}}$ in $\ell_p(\mathbb{N})$ the sequence $x + y = (x_k + y_k)_{k\in\mathbb{N}}$ satisfies

$$\|x + y\|_p = \left(\sum_{k\in\mathbb{N}} |x_k + y_k|^p \right)^{1/p} \le \left(\sum_{k\in\mathbb{N}} |x_k|^p \right)^{1/p} + \left(\sum_{k\in\mathbb{N}} |y_k|^p \right)^{1/p} = \|x\|_p + \|y\|_p. \tag{2.2.19}$$

Hence $\|x + y\|_p \le \|x\|_p + \|y\|_p < \infty$, and $x + y = (x_k + y_k)_{k\in\mathbb{N}}$ also belongs to $\ell_p(\mathbb{N})$.

We have verified that $\ell_p(\mathbb{N})$ is closed under addition and scalar multiplication, and hence $\ell_p(\mathbb{N})$ is subspace of $\ell(\mathbb{N})$ and thus itself a linear space.

For $1 < p < \infty$, we have now to verify that $\|\cdot\|_p$ satisfies the four norm properties:

(i) Since $|x_k|^p \ge 0$ for all $k \in \mathbb{N}$, we have $\|x\|_p \ge 0$ for any $x = (x_k)_{k\in\mathbb{N}}$ in $\ell_p(\mathbb{N})$.

(ii) For $x = (0)_{k\in\mathbb{N}}$, we have clearly $\|x\|_p = 0$. Now assume that for some $x = (x_k)_{k\in\mathbb{N}} \in \ell_p(\mathbb{N})$

$$0 = \|x\|_p = \left(\sum_{k\in\mathbb{N}} |x_k|^p\right)^{1/p} \qquad \Leftrightarrow \qquad 0 = \|x\|_p^p = \sum_{k\in\mathbb{N}} |x_k|^p,$$

Then we can conclude that $|x_k|^p = 0$ for all $k \in \mathbb{N}$ and hence $x_k = 0$ for all $k \in \mathbb{N}$, that is, $x = (x_k)_{k\in\mathbb{N}}$ is the zero sequence. This shows non-degeneracy.

(iii) Let $x = (x_k)_{k\in\mathbb{N}} \in \ell_p(\mathbb{N})$ and $\alpha \in \mathbb{K}$. From (2.2.18) we see then that $\|\alpha\, x\|_p = |\alpha|\, \|x\|_p$.

(iv) From (2.2.19) the triangle inequality holds: for all $x, y \in \ell_p(\mathbb{N})$,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

As $\ell_p(\mathbb{N})$ is a linear space and as $\|\cdot\|_p$ has all the properties of a norm, the space $\ell_p(\mathbb{N})$ with $\|\cdot\|_p$ is a normed linear space. □

The proofs of the next two theorems can be given analogously to the proof of Theorem 2.20 and are left as an exercise. However, for showing the non-degeneracy of the norms, you need to know about the Lebesgue integral, Lebesgue measurable functions, and sets of Lebesgue measure zero; otherwise you will not be able to verify this property.

Theorem 2.21 ($L_p([a,b])$ with $\|\cdot\|_{L_p([a,b])}$ is a normed linear space)
Let $1 \leq p < \infty$. The space $L_p([a,b])$ with the function $\|\cdot\|_{L_p([a,b])} : L_p([a,b]) \to \mathbb{R}$,

$$\|f\|_{L_p([a,b])} := \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}, \tag{2.2.20}$$

*is a **normed linear space**.*
The space $L_\infty([a,b])$ with the function $\|\cdot\|_{L_\infty([a,b])} : L_\infty([a,b]) \to \mathbb{R}$,

$$\|f\|_{L_\infty} := \operatorname*{ess-sup}_{x\in[a,b]} |f(x)|,$$

*is a **normed linear space**.*

Theorem 2.22 ($L_p(\mathbb{R})$ with $\|\cdot\|_{L_p(\mathbb{R})}$ is a normed linear space)
Let $1 \leq p < \infty$. The space $L_p(\mathbb{R})$ with the function $\|\cdot\|_{L_p(\mathbb{R})} : L_p(\mathbb{R}) \to \mathbb{R}$,

$$\|f\|_{L_p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x\right)^{1/p}, \tag{2.2.21}$$

*is a **normed linear space**.*
The space $L_\infty(\mathbb{R})$ with the function $\|\cdot\|_{L_\infty(\mathbb{R})} : L_\infty(\mathbb{R}) \to \mathbb{R}$,

$$\|f\|_{L_\infty(\mathbb{R})} := \operatorname*{ess-sup}_{x\in\mathbb{R}} |f(x)|,$$

*is a **normed linear space**.*

Exercise 20 *Show that the spaces $L_p([a,b])$, $1 < p < \infty$, with the norm $\|\cdot\|_{L_p([a,b])}$ are normed linear spaces. You do not have to show the non-degeneracy of the norm, as this requires knowledge of the Lebesgue integral.*

2.3 Open and Closed Sets, and Separable Spaces

In this section, X is always a **normed linear space with norm** $\|\cdot\| : X \to \mathbb{R}$. We mention at the beginning that, in analogy to the $\mathbb{R}^3$ with the Euclidean norm, we should think of

$$\mathrm{dist}(x,y) := \|x-y\|, \qquad x,y \in X,$$

as a measure for the **distance** between x and y. In fact, $\mathrm{dist}(x,y) := \|x-y\|$ is a **metric** or **distance function**.

Definition 2.23 (open ball, closed ball, and sphere in a normed linear space)
Let X be a normed linear space with norm $\|\cdot\| : X \to \mathbb{R}$.

(i) The ***open ball*** *centred at $x \in X$ with radius r is defined by*

$$B(x;r) := \{y \in X \,:\, \|y-x\| < r\}.$$

(ii) The ***closed ball*** *centred at $x \in X$ with radius r is defined by*

$$\widetilde{B}(x;r) := \{y \in X \,:\, \|y-x\| \le r\}.$$

(iii) The ***sphere*** *centred at $x \in X$ with radius r is defined by*

$$S(x;r) := \{y \in X \,:\, \|y-x\| = r\}.$$

Sometimes we call the open ball $B(x;r)$ an r***-neighbourhood of the point*** x.

For getting an intuition of the statements given in this and the following sections it is useful to keep the standard example of the normed linear vector space $\mathbb{R}^2$ with the Euclidean norm $\|\mathbf{x}\|_2 = (\sum_{k=1}^2 |x_k|^2)^{1/2} = \sqrt{|x_1|^2 + |x_2|^2}$ in mind, since we can easily draw pictures in this case. In fact, for getting an intuitive understanding of the concepts it is extremely useful to draw pictures to visualise the concepts whenever possible.

Example 2.24 (open and closed ball and sphere in $\mathbb{R}^2$)
Consider $\mathbb{R}^2$ endowed with the p-norms.

(a) If $p = 1$, then the open ball

$$B(0;1) = \left\{\mathbf{x} = (x_1,x_2)^T \in \mathbb{R}^2 \,:\, \|\mathbf{x}\|_1 = |x_1| + |x_2| < 1\right\}$$

is the interior of the square in the left picture in Figure 2.1. The sphere $S(0;1)$ is the boundary of the square, and the closed ball $\widetilde{B}(0;1)$ is the square including its boundary.

(b) If $p = 2$, then the open ball

$$B(0;1) = \left\{ \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2 \,:\, \|\mathbf{x}\|_2 = \sqrt{|x_1|^2 + |x_2|^2} < 1 \right\}$$

is the interior of the disc in the middle picture in Figure 2.1. The sphere $S(0;1)$ is the circle, and the closed ball $\widetilde{B}(0;1)$ is the disc including its boundary.

(c) If $p = \infty$, then the open ball

$$B(0;1) = \left\{ \mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2 \,:\, \|\mathbf{x}\|_\infty = \max\left\{|x_1|, |x_2|\right\} < 1 \right\}$$

is the interior of the square in the right picture in Figure 2.1. The sphere $S(0;1)$ is the boundary of the square, and the closed ball $\widetilde{B}(0;1)$ is the square including its boundary. □

We see in the previous example that balls in normed linear spaces are not necessarily 'round' in the geometric sense.

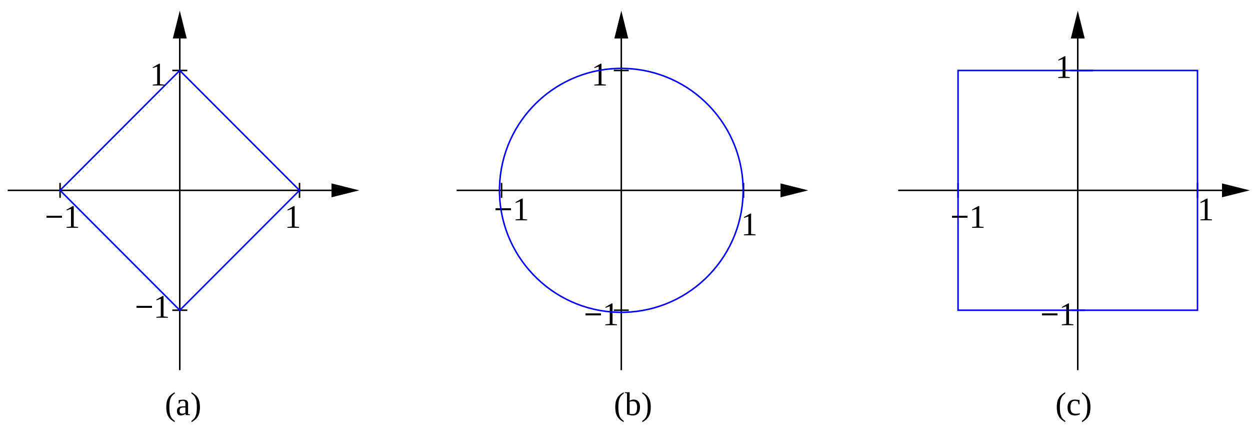

Figure 2.1: The unit ball $B(0;1)$ in $\mathbb{R}^2$ with respect to the norm $\|\cdot\|_p$, where $p = 1$ in (a), $p = 2$ in (b), and $p = \infty$ in (c).

Definition 2.25 (bounded and unbounded set)
Let X be a normed linear space. A subset $M \subset X$ is said to be ***bounded*** *if there is an $x \in X$ and a real number $r > 0$ such that $M \subset B(x;r)$. If a subset $M \subset X$ is not bounded, we call it* ***unbounded****.*

It is intuitively clear that if M is bounded, then for any $y \in X$ there exists a number r_y such that $M \subset B(y;r_y)$. Indeed, if $M \subset B(x;r_x)$, and if $y \in X$ is any other point, then from the triangle inequality

$$\|z - y\| = \|(z - x) + (x - y)\| \leq \|z - x\| + \|x - y\| < r_x + \|x - y\| =: r_y \qquad \text{for all } z \in M.$$

Hence with $r_y := r_x + \|x - y\|$, we have $M \subset B(y;r_y)$. In particular, if we choose $x = 0$, then we have the following **characterisation of a bounded set**: $M \subset X$ is bounded if there exists $r > 0$ such that $\|x\| < r$ for all $x \in M$.

Example 2.26 (bounded and unbounded sets)

(a) In any normed linear space X, open balls $B(x;r)$, closed balls $\widetilde{B}(x;r)$, and spheres $S(x;r)$ are bounded.

(b) In $\mathbb{R}^2$ with the Euclidean norm $\|\cdot\|_2$, the set $M = \{(x,0)^T \in \mathbb{R}^2 \,:\, x > 0\}$ is not bounded.

(c) In $\mathbb{R}^2$ with the Euclidean norm $\|\cdot\|_2$, the set $M = \{(t, \sin t)^T \,:\, t \in [0,\pi]\}$ is bounded.

(d) In $\mathbb{R}^2$ with the Euclidean norm $\|\cdot\|_2$, the set $M = \{(t, \sin t)^T \,:\, t \in \mathbb{R}\}$ is not bounded.

(e) In $\mathbb{C}$, with the absolute value norm $\|z\| := |z|$ the upper half plane $M = \{z \in \mathbb{C} \,:\, \Im(z) \geq 0\}$ is not bounded.

(f) The set $\mathbb{N}$ is not bounded in the real numbers $\mathbb{R}$ with the absolute value norm $\|x\| := |x|$.

(g) In the space $C(\mathbb{R})$ of continuous complex-valued functions on $\mathbb{R}$ with the supremum norm

$$\|f\|_{C(\mathbb{R})} = \sup_{x\in\mathbb{R}} |f(x)|,$$

the set $M = \{\sin(kx), \cos(mx) \,:\, k, m \in \mathbb{N}_0\}$ is bounded. □

Definition 2.27 (interior point and open and closed set)
Let X be a normed linear space with norm $\|\cdot\|$.

(i) Let M be a subset of X. A point $x \in M$ is called an ***interior point of*** *M, if there exists an $r > 0$ such that the open ball $B(x;r)$ is contained in M.*

(ii) A subset $M \subset X$ is said to be ***open*** *if every point in M is an interior point of M, that is, if for each $x \in M$ there exists an $r > 0$ (depending on x) such that the open ball $B(x;r) \subset M$.*

(iii) A subset $M \subset X$ is said to be ***closed*** *if its complement $M^c := X \setminus M$ is open.*

Example 2.28 (interior points and closed and open sets)

(a) In $\mathbb{R}$ with the absolute value norm $\|x\| := |x|$, the open interval $(a,b) = \{x \in \mathbb{R} \,:\, a < x < b\}$ is open and the closed interval $[a,b] = \{x \in \mathbb{R} \,:\, a \leq x \leq b\}$ is closed. The half-open intervals $(a,b]$ and $[a,b)$ are neither open nor closed. The interior points of (a,b), $[a,b]$, $(a,b]$ and $[a,b)$ are the same and they are all points in (a,b).

(b) In any normed linear space, an open ball $B(x;r)$ is open and a closed ball $\widetilde{B}(x;r)$ is closed. The sphere $S(x;r)$ is also closed.

(c) Any normed linear space X and the empty set $\emptyset$ are both open and closed.

(d) A straight line in $\mathbb{R}^2$, endowed with the Euclidean norm $\|\cdot\|_2$, is closed.

(e) The set of constant functions in $C([a,b])$ is closed. □

Exercise 21 *Let X be a normed linear space with norm $\|\cdot\| : X \to \mathbb{R}$. Show that the open ball*

$$B(x;r) = \{y \in X \;:|, \|y - x\| < r\}$$

is open and that the closed ball

$$\widetilde{B}(x;r) = \{y \in X \;:|, \|y - x\| \leq r\}$$

is closed.

Lemma 2.29 (union of open sets and intersection of closed sets)
Let X be a normed linear space with norm $\|\cdot\|$.

(i) The union of (finitely many or infinitely many) open sets is open.

(ii) The intersection of (finitely many or infinitely many) closed sets is closed.

Exercise 22 *Proof Lemma 2.29.*

Definition 2.30 (accumulation point)
*Let X be a normed linear space with norm $\|\cdot\|$, and let $M \subset X$. A point $x_0 \in X$ is called an **accumulation point of** M if every neighbourhood of x_0 (that is, every ball $B(x_0; r)$ centred at x_0) contains at least one point $y \in M$ distinct from x_0.*

We note that an accumulation point of a set $M \subset X$ may belong to M or not!

Definition 2.31 (closure of a subset)
*Let X be a normed linear space with norm $\|\cdot\|$, and let $M \subset X$. The set consisting of the points of M and the accumulation points of M is called the **closure of** M and is denoted by $\overline{M}$.*

Example 2.32 (accumulation points and closure)

(a) Consider $\mathbb{R}$ with the absolute value norm $\|x\| := |x|$. The set of accumulation points of the intervals (a, b), $[a, b]$, $(a, b]$, and $[a, b)$ is the same and is given by $[a, b]$. So the closure of each of these intervals is $[a, b]$.

(b) Let X be a normed linear space. The set of accumulation points of the open ball $B(x; r)$ is the closed ball $\widetilde{B}(x; r)$, and this closed ball is also the closure of $B(x; r)$.

(c) The set of integers $\mathbb{N}$ as a subset of $\mathbb{R}$ with the absolute value norm $\|x\| := |x|$ is closed. $\mathbb{N}$ has no interior points and no accumulation points. The closure of $\mathbb{N}$ is $\mathbb{N}$ itself. □

Theorem 2.33 (characterisation of the closure of a set)
Let X be a normed linear space with norm $\|\cdot\|$, and let $M \subset X$. The closure $\overline{M}$ of M is the smallest closed set containing M. In other words, for any closed set A with $M \subset A$ one has $\overline{M} \subset A$. Equivalently,

$$\overline{M} = \bigcap_{\substack{M \subset A, \\ A \subset X \text{ is closed}}} A. \tag{2.3.1}$$

We note that, from Lemma 2.29, it is clear that the set on the right-hand side of (2.3.1) is closed.

Proof of Theorem 2.33: We need to prove first of all that $\overline{M}$ is closed. We shall do it in two steps.

Step 1: First we show that $\overline{M}$ contains all its accumulation points. Let $x_0 \in X$ be an accumulation point of $\overline{M}$. Then by definition for any $\epsilon > 0$ there is a point $y \in \overline{M}$ distinct from x_0 such that $\|x_0 - y\| < \epsilon/2$. By definition of $\overline{M}$ the point y either belongs to the set M, in which case we set $z = y$, or y is an accumulation point of M. In the latter case there exists a $z \in M$ distinct from y such that $\|z - y\| < \|x_0 - y\|/2$, and therefore from the triangle inequality

$$\|x_0 - z\| \leq \|x_0 - y\| + \|y - z\| < \|x_0 - y\| + \frac{1}{2}\,\|x_0 - y\| = \frac{3}{2}\,\|x_0 - y\| < \frac{3\,\epsilon}{4}.$$

Note that z is distinct from x_0 since from the lower triangle inequality

$$\|x_0 - z\| \geq \big|\|x_0 - y\| - \|y - z\|\big| \geq \|x_0 - y\| - \|y - z\| \geq \|x_0 - y\| - \frac{1}{2}\,\|x_0 - y\| = \frac{1}{2}\,\|x_0 - y\| > 0.$$

Thus for a given ϵ we have found a vector $z \in M$ distinct from x_0 such that $z \in B(x_0; 3\epsilon/4)$. This shows that x_0 is an accumulation point of M and therefore belongs to $\overline{M}$. Thus M contains all its accumulation points.

Step 2: Now we prove that $\overline{M}$ is closed, or, which is the same, that the complement $\overline{M}^c = X \setminus \overline{M}$ is open. Suppose that it is not true, that is, suppose that $\overline{M}^c$ is not open. Then there is a $x_0 \in \overline{M}^c$ such that for any $\epsilon > 0$ one can find a vector $y \in B(x_0, \epsilon)$ such that $y \notin \overline{M}^c$. Since $y \in \overline{M}$ this implies that x_0 is an accumulation point of $\overline{M}$ and by Step 1 must belong to $\overline{M}$. This contradicts the assumption $x_0 \in \overline{M}^c$. Therefore our assumption that $\overline{M}^c$ was not open is wrong, and we have verified that $\overline{M}^c$ is open and $\overline{M}$ is closed.

Now we are in a position to complete the proof of the theorem. Since $\overline{M}$ is closed and contains M, we obviously have

$$\left(\bigcap_{\substack{M \subset A, \\ A \subset X \text{ is closed}}} A \right) \subset \overline{M}.$$

Suppose that the other inclusion $\supset$ is not true. Then for some closed set A with $M \subset A$ there is a point $x_0 \in \overline{M} \setminus M$ such that $x_0 \notin A$. Then $x_0 \in A^c$. As the set A^c is open, there is a number $\epsilon > 0$ such that $B(x_0; \epsilon) \subset A^c$. Remembering that $M \subset A$ and hence $A^c \subset M^c$, we conclude that $B(x_0; \epsilon) \subset M^c$ as well. Therefore x_0 cannot be an accumulation point of M which contradicts the fact that $x_0 \in \overline{M} \setminus M$. Hence our assumption was wrong and the inclusion $\supset$ is also true. □

The next definition is very important.

Definition 2.34 (dense subset and separable normed linear space)
Let X be a normed linear space with norm $\|\cdot\|$.

(i) *A subset M of X is said to be* ***dense in*** *X if $\overline{M} = X$.*

(ii) *The space X is said to be* ***separable*** *if it has a countable subset M that is dense in X. (A set $M \subset X$ is countable, if there exists a one-to-one correspondence between the elements of M and the positive integers.)*

From the definitions given above, we can conclude that, if M is dense in X, then every element in X is either in M or is an accumulation point of M. Hence for every $x \in X$ every neighbourhood $B(x;r)$ will contain at least one element of M.

Example 2.35 (separable spaces)

(a) The space $\mathbb{R}$ with $\|x\| := |x|$ is separable, because the set of rational numbers $\mathbb{Q}$ is countable and is dense in $\mathbb{R}$.

(b) The set of numbers whose real and imaginary parts are both rational, is countable and dense in $\mathbb{C}$ with the absolute value norm $\|z\| := |z|$. Therefore $\mathbb{C}$ with the absolute value norm $\|z\| := |z|$ is separable.

(c) The Euclidean space $\mathbb{R}^d$ with the Euclidean norm $\|\cdot\|_2$ is separable, because the space $\mathbb{Q}^d$ is countable and dense in $\mathbb{R}^d$. □

Exercise 23 *Show that $\mathbb{R}$ with the absolute value norm $\|x\| := |x|$ is separable.*

Checking whether a space is separable or not is not always trivial, and we will encounter more complicated examples in the next two lemmas.

Lemma 2.36 ($\ell_p(\mathbb{N})$ is separable for $1 \le p < \infty$)
For p satisfying $1 \le p < \infty$, the sequence space $\ell_p(\mathbb{N})$ with the norm

$$\|x\|_p = \left(\sum_{k\in\mathbb{N}} |x_k|^p\right)^{1/p}, \qquad x = (x_k)_{k\in\mathbb{N}},$$

is separable.

Proof of Lemma 2.36: The proof is given in three steps.

Step 1: We first show that the set of all finite sequences is dense in $\ell_p(\mathbb{N})$. Let M be the subset of $\ell_p(\mathbb{N})$ consisting of the sequences of the form

$$x = (x_1, x_2, \dots, x_n, 0, 0, \dots), \qquad n \in \mathbb{N}.$$

We need to show that for every fixed element $y = (y_k)_{k\in\mathbb{N}} \in \ell_p(\mathbb{N})$ and every $\epsilon > 0$ the open ball $B(y;\epsilon/2)$ contains at least one element from M. For $n \in \mathbb{N}$, define $y^{(n)} \in M$ as follows:

$$y^{(n)} := (y_1, y_2, \dots, y_n, 0, 0 \dots).$$

Then

$$y - y^{(n)} = (0, 0, \dots, 0, y_{n+1}, y_{n+2}, \dots).$$

As $y \in \ell_p(\mathbb{N})$, we have

$$\|y\|_p = \left(\sum_{k\in\mathbb{N}} |y_k|^p\right)^{1/p} < \infty,$$

that is, the infinite sum in the parentheses converges. Hence for every $\epsilon > 0$ there exists and $N = N(\epsilon) \in \mathbb{N}$ such that

$$\left(\sum_{k=n+1}^{\infty} |y_k|^p\right)^{1/p} < \frac{\epsilon}{2} \qquad \text{for all } n \geq N,$$

and thus

$$\|y - y^{(n)}\|_p = \left(\sum_{k=n+1}^{\infty} |y_k|^p\right)^{1/p} < \frac{\epsilon}{2} \qquad \text{for all } n \geq N.$$

In particular, we have for $y^{(N)} \in M$ the estimate

$$\|y - y^{(N)}\| < \frac{\epsilon}{2}. \tag{2.3.2}$$

Step 2: Let an arbitrary $y = (y_k)_{k\in\mathbb{N}}$ be chosen from $\ell_p(\mathbb{N})$ and let $y^{(N)}$ be constructed as in Step 1 such that (2.3.2) holds true. Then we can approximate every non-zero component y_k, $k = 1, 2, \ldots, N$ of the element $y^{(N)}$ by a rational number z_k in such a way that $|y_k - z_k| < (\epsilon/2)/N^{1/p}$ for $k = 1, 2, \ldots, N$. Then for $z = (z_1, z_2, \ldots, z_N, 0, 0, \ldots)$

$$\|y^{(N)} - z\|_p = \left(\sum_{k\in\mathbb{N}} |y_k^{(N)} - z_k|^p\right)^{\frac{1}{p}} = \left(\sum_{k=1}^{N} |y_k - z_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{N} \frac{(\epsilon/2)^p}{N}\right)^{\frac{1}{p}} = \frac{\epsilon}{2}. \tag{2.3.3}$$

Step 3: Now use the triangle in equality (that is, the Minkowski inequality for $\ell_p(\mathbb{N})$) to check that, from (2.3.2) and (2.3.3),

$$\|y - z\|_p = \|(y - y^{(N)}) + (y^{(N)} - z)\|_p \leq \|y - y^{(N)}\|_p + \|y^{(N)} - z\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

As $y \in \ell_p(\mathbb{N})$ and $\epsilon > 0$ were arbitrary, we have shown that

$$A := \bigcup_{n=1}^{\infty} \{z = (z_1, z_2, \ldots, z_n, 0, 0, \ldots) \, : \, z_j \in \mathbb{Q}\}$$

is dense in $\ell_p(\mathbb{N})$. Since $\mathbb{Q}^n$ is countable, the set A is countable. Hence $\ell_p(\mathbb{N})$ is separable. □

Lemma 2.37 ($\ell_\infty(\mathbb{N})$ is not separable)
The sequence space $\ell_\infty(\mathbb{N})$ with the norm

$$\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|, \qquad x = (x_k)_{k\in\mathbb{N}}.$$

is not separable.

Exercise 24 *Prove Lemma 2.37. (Hint: Give a proof by contradiction. Assume that $\ell_\infty(\mathbb{N})$ was separable. Then there exists a countable dense subset $M = \{x^{(1)}, x^{(2)}, \ldots\}$. Now use this set M to construct an element in $\ell_\infty(\mathbb{N})$ which is not in $\overline{M}$.)*

2.4 Convergence and Completeness

In this section we introduce the notions of convergence of sequences and completeness of a normed linear space. Below $(x_n)_{n\in\mathbb{N}}$, where $x_n \in X$, denotes a sequence in the space X. (Do not confuse the notation with the elements of the $\ell_p(\mathbb{N})$ spaces.)

Definition 2.38 (convergent/divergent sequence and limit)
Let X be a normed linear space with norm $\|\cdot\| : X \to \mathbb{K}$.

(i) A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is said to ***converge*** *(or* ***to be convergent****) if there exists an $x \in X$ such that*

$$\lim_{n\to\infty} \|x - x_n\| = 0. \tag{2.4.1}$$

(Equivalently, $(x_n)_{n\in\mathbb{N}} \subset X$ converges if for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that $\|x_n - x\| < \epsilon$ for all $n \geq N$.) The element x in (2.4.1) is then called the ***limit*** *of the sequence $(x_n)_{n\in\mathbb{N}}$. We also write $x_n \to x$ as $n \to \infty$ or $x = \lim_{n\to\infty} x_n$, and we say $(x_n)_{n\in\infty}$* ***converges to*** *x.*

(ii) A sequence is said to be ***divergent*** *if it does not converge.*

We note some consequences:

From the lower triangle inequality

$$\big|\|x_n\| - \|x\|\big| \leq \|x_n - x\|;$$

hence if $x_n \to x$ for $n \to \infty$, then also $\|x_n\| \to \|x\|$ for $n \to \infty$.

If $x_n \to x$ and $y_n \to y$ for $n \to \infty$, then $(x_n + y_n) \to x + y$ for $n \to \infty$. Indeed,

$$\big\|(x_n + y_n) - (x + y)\big\| = \|(x_n - x) + (y_n - y)\| \leq \|x_n - x\| + \|y_n - y\| \to 0 \qquad \text{as } n \to \infty.$$

Definition 2.39 (Cauchy sequence)
Let X be a normed linear space with norm $\|\cdot\| : X \to \mathbb{K}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be a ***Cauchy sequence*** *if for every $\epsilon > 0$ there exists an $N = N(\epsilon) \in \mathbb{N}$ such that*

$$\|x_n - x_m\| < \epsilon \qquad \text{for all } m, n \geq N.$$

Example 2.40 (convergent sequence in $\mathbb{R}^3$)
In $\mathbb{R}^3$ with the Euclidean norm $\|\mathbf{x}\|_2 = (\sum_{k=1}^{3} |x_k|^2)^{1/2}$ the sequence $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$, defined by $\mathbf{x}^{(n)} = (1/n, e^{1/n}, 2)^T$ converges to $\mathbf{x} = (0, 1, 2)^T$. Indeed

$$\|\mathbf{x}^{(n)} - \mathbf{x}\|_2 = \|(1/n, e^{1/n} - 1, 0)^T\|_2 = \sqrt{(1/n)^2 + (e^{1/n} - 1)^2 + 0}^{\,1/2} \to 0 \qquad \text{as } n \to \infty.$$

It is easily checked that this sequence is also a Cauchy sequence. □

Example 2.41 (convergent sequence in $C([a,b])$)
The sequence $(f_n)_{n\in\mathbb{N}}$, defined by $f_n(x) := \exp(x + x/n)$ in $C([0,1])$ with the supremum norm

$$\|f\|_{C([0,1])} := \sup_{x\in[0,1]} |f(x)|$$

converges uniformly on $[0,1]$ to the function $f(x) := \exp(x)$. Indeed,

$$0 \le |f_n(x) - f(x)| = \left|e^{x+x/n} - e^x\right| = \left|e^x\,(e^{x/n} - 1)\right| \le e\,(e^{1/n} - 1) \qquad \text{for all } x \in [0,1].$$

As $e\,(e^{1/n} - 1) \to 0$ as $n \to \infty$, we find from the sandwich theorem that

$$0 \le \lim_{n\to\infty} \sup_{x\in[0,1]} |f_n(x) - f(x)| \le \lim_{n\to\infty} e\,(e^{1/n} - 1) = 0,$$

and hence $\lim_{n\to\infty} \|f_n - f\|_{C([0,1])} = 0$. □

The notions of accumulation points and closure introduced in the previous section can also be described using the notion of convergent sequences. For example, the following is equivalent to Definitions 2.30 and 2.34:

Lemma 2.42 (accumulation point, dense subset characterised with sequences)
Let X be a normed linear space with norm $\|\cdot\| : X \to \mathbb{K}$.

(i) A point $x_0 \in X$ is an ***accumulation point*** *of a subset M if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in M such that $x_n \neq x_0$ and $x_n \to x_0$ as $n \to \infty$.*

(ii) A subset M is ***dense in*** *X if for each $x \in X$ there is a sequence $(x_n)_{n\in\mathbb{N}}$ in M such that $x_n \to x$ as $n \to \infty$.*

Exercise 25 *Prove Lemma 2.42.*

Lemma 2.43 (properties of convergent sequences)
Let X be a normed linear space with norm $\|\cdot\| : X \to \mathbb{K}$. Let $(x_n)_{n\in\mathbb{N}}$ be a ***convergent sequence in*** *X. Then the following holds true:*

(i) The sequence $(x_n)_{n\in\mathbb{N}}$ is ***bounded****, that is, there exists $r > 0$ such that $\|x_n\| \le r$ for all $n \in \mathbb{N}$ (or equivalently, there exists some $y \in X$ and some $r_y > 0$ such that $x_n \in B(y; r_y)$ for all $n \in \mathbb{N}$).*

(ii) The limit of $(x_n)_{n\in\mathbb{N}}$ is ***unique****.*

(iii) The sequence $(x_n)_{n\in\mathbb{N}}$ is a ***Cauchy sequence****.*

Proof of Lemma 2.43: Let $x \in X$ be a limit of the convergent sequence $(x_k)_{k\in\mathbb{N}}$, that is, $\lim_{n\to\infty} \|x_n - x\| = 0$.

(i) By the definition of convergence, for any $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $\|x - x_n\| < \epsilon$ for all $n \ge N$. Thus

$$\|x - x_n\| \le \max\left\{\epsilon, \max_{m=1,\dots,N-1} \|x - x_m\|\right\}, \qquad \text{for all } n \in \mathbb{N},$$

and hence $x_n \in B(x;r)$ with

$$r := \max\left\{\epsilon, \max_{m=1,\dots,N-1} \|x - x_m\|\right\} + \frac{1}{2}.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is bounded.

(ii) To prove uniqueness of the limit, suppose that there are two limits x and $\widetilde{x}$. Given $\epsilon > 0$, there exist $N_1 = N_1(\epsilon)$ and $N_2 = N_2(\epsilon)$ such that

$$\|x_n - x\| < \frac{\epsilon}{2} \quad \text{for all } n \geq N_1 \qquad \text{and} \qquad \|x_n - \widetilde{x}\| < \frac{\epsilon}{2} \quad \text{for all } n \geq N_2.$$

Thus with $N = \max\{N_1, N_2\}$,

$$\|x_n - x\| < \frac{\epsilon}{2} \qquad \text{and} \qquad \|x_n - \widetilde{x}\| < \frac{\epsilon}{2} \qquad \text{for all } n \geq N.$$

Consequently, from the triangle inequality

$$0 \leq \|x - \tilde{x}\| = \|(x - x_n) + (x_n - \widetilde{x})\| \leq \|x - x_n\| + \|x_n - \widetilde{x}\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for all } n \geq N.$$

As $\epsilon > 0$ was arbitrary, this shows that $x = \tilde{x}$. Hence the limit is unique.

(iii) By the definition of convergence, for every $\epsilon > 0$ there exists an $N = N(\epsilon) \in \mathbb{N}$ such that $\|x - x_n\| < \epsilon/2$ for all $n \geq N$. Thus, using the triangle inequality,

$$\|x_n - x_m\| = \|(x_n - x) + (x - x_m)\| \leq \|x_n - x\| + \|x_m - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for all } m, n \geq N.$$

This shows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

This concludes the proof. □

Exercise 26 *Let X be a normed linear space with norm $\|\cdot\| : X \to \mathbb{R}$. Show that every Cauchy sequence in X is bounded.*

By Lemma 2.43 every convergent sequence $(x_n)_{n\in\mathbb{N}}$ in a normed linear space is a Cauchy sequence. It is also easily shown that every Cauchy sequence is bounded. However, **it is in general not true that every Cauchy sequence is convergent!** Whether this is true (or not) depends on the particular normed linear space.

Definition 2.44 (complete normed linear space = Banach space)
A normed space X is said to be ***complete*** *if every Cauchy sequence in X is convergent. A complete normed linear space is also called a* ***Banach space****.*

Theorem 2.45 ($\ell_p(\mathbb{N})$ is complete for $1 \leq p \leq \infty$)
Let $1 \leq p \leq \infty$. The space $\ell_p(\mathbb{N})$, with the p-norm (2.2.16) for $1 \leq p < \infty$ and (2.2.17) for $p = \infty$, is complete, that is, it is a Banach space.

Proof of Theorem 2.45: We give the proof only for the case that $1 \le p < \infty$. The case $p = \infty$ is left as an exercise.

Let $1 \le p < \infty$. Let $(x^{(n)})_{n\in\mathbb{N}}$, where $x^{(n)} := (x_k^{(n)})_{k\in\mathbb{N}}$ be an arbitrary Cauchy sequence in $\ell_p(\mathbb{N})$. Then for any $\epsilon > 0$ there exists a number $N = N(\epsilon) \in \mathbb{N}$ such that

$$\|x^{(n)} - x^{(m)}\|_p < \epsilon, \qquad \text{for all } n, m \ge N. \tag{2.4.2}$$

For every $j \in \mathbb{N}$ the sequence $(x_j^{(n)})_{n\in\mathbb{N}}$ of jth entries of the $x^{(n)} = (x_k^{(n)})_{k\in\mathbb{N}}$ is a Cauchy sequence of numbers in $\mathbb{K}$ because (2.4.2) implies that for $1 \le p < \infty$

$$|x_j^{(n)} - x_j^{(m)}| = \left(|x_j^{(n)} - x_j^{(m)}|^p\right)^{1/p} \le \left(\sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p\right)^{1/p} = \|x^{(n)} - x^{(m)}\|_p < \epsilon \quad \text{for all } m, n \ge N.$$

Since $\mathbb{K} = \mathbb{C}$ and $\mathbb{K} = \mathbb{R}$ with the absolute value norm $\|\cdot\| := |\cdot|$ are complete, the Cauchy sequence $(x_j^{(n)})_{n\in\mathbb{N}}$ has a limit $x_j = \lim_{n\to\infty} x_j^{(n)}$ in $\mathbb{K}$. It remains to show that the element $x := (x_1, x_2, \ldots, x_j, \ldots) = (x_j)_{j\in\mathbb{N}}$ belongs to $\ell_p(\mathbb{N})$ and that x is the limit of the $\ell_p(\mathbb{N})$ Cauchy sequence $(x^{(n)})_{n\in\mathbb{N}}$. Then we have shown that the Cauchy sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ converges in $\ell_p(\mathbb{N})$. Since $(x^{(n)})_{n\in\mathbb{N}}$ was an arbitrary Cauchy sequence in $\ell_p(\mathbb{N})$, this shows that $\ell_p(\mathbb{N})$ is complete.

We start by showing $x = (x_j)_{j\in\mathbb{N}} \in \ell_p(\mathbb{N})$. Since a Cauchy sequence is bounded, there exists an $M > 0$ such that $\|x^{(n)}\|_p \le M$ for all $n \in \mathbb{N}$. Hence, we have

$$\left(\sum_{j=1}^{k} |x_j^{(n)}|^p\right)^{1/p} \le \|x^{(n)}\|_p \le M \qquad \text{for all } k \in \mathbb{N} \textit{ and } \text{all } n \in \mathbb{N}.$$

Since this estimate is uniform in $k \in \mathbb{N}$ and $n \in \mathbb{N}$, and since $\lim_{n\to\infty} x_j^{(n)} = x_j$ for all $j \in \mathbb{N}$, we can first let n tend to infinity to derive

$$\left(\sum_{j=1}^{k} |x_j|^p\right)^{1/p} = \lim_{n\to\infty} \left(\sum_{j=1}^{k} |x_j^{(n)}|^p\right)^{1/p} \le M \qquad \text{for all } k \in \mathbb{N}.$$

The sequence $(s_k)_{k\in\mathbb{N}}$ of real numbers $s_k := \left(\sum_{j=1}^{k} |x_j|^p\right)^{1/p}$ is increasing and bounded from above by M. Hence we know that it converges and that the limit is also bounded by M, that is

$$\|x\|_p = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} = \lim_{k\to\infty} \left(\sum_{j=1}^{k} |x_j|^p\right)^{1/p} \le M.$$

Thus $x = (x_j)_{j\in\mathbb{N}}$ is in $\ell_p(\mathbb{N})$.

Finally, to see that $(x^{(n)})_{n\in\mathbb{N}}$ converges to x, we have to show that for every $\epsilon > 0$ there exists an $N = N(\epsilon)$ such that $\|x - x^{(n)}\|_p < \epsilon$ for $n \ge N$. As $(x^{(n)})_{n\in\mathbb{N}}$ is a Cauchy sequence, given

$\epsilon > 0$, there exists an $N = N(\epsilon)$ such that $\|x^{(n)} - x^{(m)}\|_p < \epsilon$ for all $n, m \geq N$. This means in particular, that we have for any $k \in \mathbb{N}$:

$$\left(\sum_{j=1}^{k} |x_j^{(n)} - x_j^{(m)}|^p \right)^{1/p} \leq \|x^{(n)} - x^{(m)}\|_p \leq \epsilon \qquad \text{for all } n, m \geq N.$$

Keeping n and k fixed and letting m tend to infinity yields

$$\left(\sum_{j=1}^{k} |x_j^{(n)} - x_j|^p \right)^{1/p} \leq \epsilon \qquad \text{for all } n \geq N \text{ and all } k \in \mathbb{N},$$

because $\lim_{n\to\infty} x_j^{(n)} = x_j$ for $j = 1, 2, \ldots, k$. Since this holds uniformly for all $k \in \mathbb{N}$ we can let k tend to infinity to find $\|x^{(n)} - x\|_p \leq \epsilon$ for $n \geq N$, which shows that $(x^{(n)})_{n\in\mathbb{N}}$ converges to x. □

We close this section by giving some more examples.

Example 2.46 ($C([a, b])$ with the supremum norm is complete)
Let $C([a, b])$ be the space of continuous complex-valued functions on the closed interval $[a, b]$ with the supremum norm

$$\|f\|_{C([a,b])} := \sup_{x\in[a,b]} |f(x)|.$$

Convergence in this space is **uniform convergence on** $[a, b]$, and the space $C([a, b])$ with $\|\cdot\|_{C([a,b])}$ is complete. This is not trivial to show; the crucial point is to show that any uniform Cauchy sequence of continuous functions converges uniformly on $[a, b]$ to a continuous function. (See 'Further Analysis‘ for details.) □

Example 2.47 ($C([a, b])$ with norm $\|\cdot\|_{L_p}$, $1 \leq q < \infty$ is not complete)
Let $C([a, b])$ be the space of functions continuous complex-valued functions on the closed interval $[a, b]$ endowed with the norm

$$\|f\|_{L_p([a,b])} = \left(\int_a^b |f(x)|^p \, \mathrm{d}x \right)^{1/p},$$

where $1 \leq p < \infty$. This space is not complete! To show this, find a sequence of continuous functions that is a Cauchy sequence with respect to $\|\cdot\|_{L_p([a,b])}$ but whose limit is not in $C([a, b])$, that is, whose limit is not continuous. □

The last two examples show that the notion of completeness depends on both the space and on the definition of the norm. To achieve completeness in Example 2.47 above, we have to extend the space $C([a, b])$ to the larger space $L_p([a, b])$.

Exercise 27 *Show that the linear space $C([0, 2])$ of continuous complex-valued functions with the norm*

$$\|f\|_{L_1([0,2])} = \int_0^2 |f(x)| \, \mathrm{d}x$$

is not complete. (Hint: Consider the following sequence of continuous functions: $(f_n)_{n\in\mathbb{N}}$

$$f_n(x) := \begin{cases} x^n & \text{if} \quad x \in [0,1], \\ 1 & \text{if} \quad x \in (1,2]. \end{cases}$$

Show that this sequence is a Cauchy sequence with respect to the norm $\|\cdot\|_{L_1([0,2])}$. Find the pointwise limit of this sequence and show that the sequence converges in the $\|\cdot\|_{L_1([0,2])}$ norm to the pointwise limit. Draw some conclusions.)

Exercise 28 *Show that the sequence space $\ell_\infty(\mathbb{N})$ with the norm (2.2.17) is complete.*

Theorem 2.48 ($L_p([a,b])$ and $L_p(\mathbb{R})$, where $1 \le p < \infty$, are complete)
Let $1 \le p < \infty$.

(i) The space $L_p([a,b])$, endowed with the norm $\|\cdot\|_{L_p([a,b])}$, defined by (2.2.20), is complete.

(ii) The space $L_p(\mathbb{R})$, endowed with the norm $\|\cdot\|_{L_p(\mathbb{R})}$, defined by (2.2.21), is complete.

(iii) The set of continuous functions is dense in $L_p([a,b])$.

The proof of this lemma is non-trivial and demands a deep knowledge of the Lebesgue integral.

Exercise 29 *Consider the vector space $\Pi([0,1])$ of all polynomials on the interval $[0,1]$ with real coefficients, endowed with the supremum norm*

$$\|f\|_{C[0,1]} = \sup_{x\in[0,1]} |f(x)|.$$

Is this space complete or not? Give a proof of your answer! (Hint: Make use of your knowledge about the convergence of power series.)

Exercise 30 *Let X be a normed linear space and let M be a closed subset of X. Show that any $x \in X \setminus M$ has non-zero distance from M, that is,*

$$\text{dist}(x, M) := \inf_{y\in M} \|x - y\| > 0.$$

Chapter 3

Inner Product Spaces

Inner product spaces are a special case of normed linear spaces, which have a norm that is induced by an **inner product**. In particular, complete inner product spaces are called **Hilbert spaces** and these are the main concern of this chapter. Examples of Hilbert spaces are $\mathbb{R}^n$ and $\mathbb{C}^n$, and $\ell_2(\mathbb{N})$ and $L_2([a,b])$, $L_2(\mathbb{R})$, each endowed with an appropriate inner product, of course.

The concept of an inner product space, which is familiar from linear algebra, will in this course primarily be used for **infinite-dimensional spaces**, namely the sequence space $\ell_2(\mathbb{N})$ and the function spaces $L_2([a,b])$ and $L_2(\mathbb{R})$. For these spaces we will introduce a **countable orthonormal basis**. In this chapter we encounter two orthonormal bases as examples, which will play a crucial role in this course: (a) the **complex trigonometric trigonometric basis polynomials** which provide an orthonormal basis for the space $L_2([-\pi,\pi])$ and (b) the **Haar scaling function** and the **Haar wavelet** which will later-on be used to construct an orthonormal basis for $L_2(\mathbb{R})$.

In Section 3.1 we introduce **inner product spaces** and discuss their basic properties and the concept of **orthogonality**. An inner product space is, in particular, also a normed linear space with a norm that is induced by an inner product. This means that all the terminology discussed in the last chapter for normed linear spaces applies to inner product spaces as well. A **complete** inner product space is called a **Hilbert space**.

In Section 3.2 we discuss the concepts of **distance**, **best approximation**, and **(orthogonal) projection**. In Section 3.3 we return to the concept of orthogonality and consider orthonormal sets in a Hilbert space. In particular, we will focus on **infinite countable orthonormal sets** in an infinite-dimensional inner product space.

In Section 3.4 we introduce the concept of a **Schauder basis** of an infinite dimensional normed linear space. If we have a Hilbert space and if such a Schauder basis consists of orthonormal elements, then the inner product space has an **orthonormal (Schauder) basis** (or a **complete orthonormal set**). An orthonormal set $M \subset H$ is an orthonormal basis in a Hilbert space H if $\operatorname{span} M$ is dense in H. We will see that an orthonormal basis has many beautiful and useful properties.

3.1 Definitions and Properties of Inner Product Spaces

This section focusses on concepts that are known from linear algebra: an inner product for a linear space, the norm induced by an inner product and orthogonality. While this appears to be nothing new, the difference to linear algebra is that we will apply these concepts for infinite-dimensional spaces and (infinite-dimensional) spaces of functions. You should make sure that you do the exercises to familiarise yourself with this framework.

Definition 3.1 (inner product and complex inner product space)
*Let X be a complex linear space over $\mathbb{K} = \mathbb{C}$ with the vector addition $\oplus$ and the scalar multiplication $\odot$. An **inner product** (or **scalar product**) on X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$, which associates to every ordered pair of elements $x, y \in X$ a scalar in $\mathbb{K} = \mathbb{C}$, possessing the following properties:*

(i) Linearity:

$$\langle x \oplus y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \qquad \text{for all } x, y, z, \in X. \tag{3.1.1}$$

(ii) Homogeneity:

$$\langle \alpha \odot x, y \rangle = \alpha \, \langle x, y \rangle \qquad \text{for all } x, y \in X \text{ and all } \alpha \in \mathbb{C}. \tag{3.1.2}$$

(iii) Anti-symmetry:

$$\langle x, y \rangle = \overline{\langle y, x \rangle} \qquad \text{for all } x, y \in X. \tag{3.1.3}$$

(iv) Non-degeneracy:

$$\langle x, x \rangle \geq 0, \qquad \text{and} \qquad \langle x, x \rangle = 0 \text{ if and only if } x = 0. \tag{3.1.4}$$

*A complex linear space X with an inner product $\langle \cdot, \cdot \rangle$ is called a complex **inner product space** (or a complex **pre-Hilbert space**).*

Combining the properties (3.1.1), (3.1.2) and (3.1.3), we see that for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{C}$

$$\langle \alpha \odot x \oplus \beta \odot y, z \rangle = \alpha \, \langle x, z \rangle + \beta \, \langle y, z \rangle, \tag{3.1.5}$$

$$\langle x, \alpha \odot y \oplus \beta \odot z \rangle = \overline{\alpha} \, \langle x, y \rangle + \overline{\beta} \, \langle x, z \rangle. \tag{3.1.6}$$

Due to these properties the inner product is said to be **sesqui-linear**, which means '$1\frac{1}{2}$-linear'.

Here is an example of a complex inner product space.

Example 3.2 (complex inner product space $\mathbb{C}^d$)
The complex linear space $\mathbb{C}^d$ with the inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 := \sum_{k=1}^{d} x_k \, \overline{y_k}, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_d)^T, \ \mathbf{y} = (y_1, y_2, \ldots, y_d)^T \in \mathbb{C}^d, \tag{3.1.7}$$

is a complex inner product space. □

In a **real** linear space over the field $\mathbb{K} = \mathbb{R}$ with an inner product the conjugation is obsolete and we obtain the following definition.

Definition 3.3 (inner product and real inner product space)
*Let X be a real linear space over $\mathbb{K} = \mathbb{R}$ with the vector addition $\oplus$ and the scalar multiplication $\odot$. An **inner product** (or **scalar product**) on X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$, which associates to every ordered pair of elements $x, y \in X$ a scalar in $\mathbb{K} = \mathbb{R}$, possessing the following properties:*

(i) Linearity:

$$\langle x \oplus y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \qquad \text{for all } x, y, z, \in X. \tag{3.1.8}$$

(ii) Homogeneity:

$$\langle \alpha \odot x, y \rangle = \alpha \langle x, y \rangle \qquad \text{for all } x, y \in X \text{ and all } \alpha \in \mathbb{R}. \tag{3.1.9}$$

(iii) Symmetry:

$$\langle x, y \rangle = \langle y, x \rangle \qquad \text{for all } x, y \in X. \tag{3.1.10}$$

(iv) Non-degeneracy:

$$\langle x, x \rangle \geq 0, \qquad \text{and} \qquad \langle x, x \rangle = 0 \text{ if and only if } x = 0. \tag{3.1.11}$$

*A real linear space X with an inner product $\langle \cdot, \cdot \rangle$ is called a real **inner product space** (or a real **pre-Hilbert space**).*

Combining the properties (3.1.8), (3.1.9) and (3.1.10), we see that for all $x, y \in X$ and all $\alpha, \beta \in \mathbb{R}$

$$\langle \alpha \odot x \oplus \beta \odot y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle, \tag{3.1.12}$$

$$\langle x, \alpha \odot y \oplus \beta \odot z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle. \tag{3.1.13}$$

Due to these properties the inner product is said to be **bi-linear**, which means 'linear in each argument'.

Here is an example of a real inner product space.

Example 3.4 (real inner product space $\mathbb{R}^d$)
The real linear space $\mathbb{R}^d$ with the Euclidean inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 := \sum_{k=1}^{d} x_k \, y_k, \qquad \mathbf{x} = (x_1, x_2, \ldots, x_d)^T, \ \mathbf{y} = (y_1, y_2, \ldots, y_d)^T \in \mathbb{R}^d, \tag{3.1.14}$$

is a real inner product space. □

Exercise 31 *Show that the function*

$$\langle \mathbf{x}, \mathbf{y} \rangle := (x_1, x_2) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where $\mathbf{x} = (x_1, x_2)^T$, $\mathbf{y} = (y_1, y_2)^T$, *defines an inner product for the real linear space* $\mathbb{R}^2$.

Notation: To ease notation, we will from now on write '+' for the vector addition '$\oplus$', and we will write '$\cdot$', or even omit the '$\cdot$', for the scalar multiplication '$\odot$'. It should be kept in mind that these operations need not be addition and scalar multiplication in the sense usually implied by the symbols '+' and '$\cdot$'. Likewise if we write '$x - y$' we mean '$x \oplus ((-1) \odot y)$'.

Let X be an inner product space with inner product $\langle \cdot, \cdot \rangle$. Then the space X has a **natural norm** which is **induced by the inner product**:

$$\|x\| := \sqrt{\langle x, x \rangle}, \qquad x \in X. \tag{3.1.15}$$

One can check easily that this norm obeys all the properties of Definition 2.1 of a norm. However, to prove that (3.1.15) obeys the triangle inequality we need the so-called **Cauchy-Schwarz inequality**.

Lemma 3.5 (Cauchy-Schwarz inequality)
Let X *be a (real or complex) inner product space with the inner product* $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$, *and let* $\|\cdot\| : X \to \mathbb{R}$ *be defined by* $\|x\| := \sqrt{\langle x, x \rangle}$. *Then the* ***Cauchy-Schwarz inequality*** *holds*

$$|\langle x, y \rangle| \leq \|x\| \, \|y\| \qquad \text{for all } x, y \in X. \tag{3.1.16}$$

Equality holds in (3.1.16) if and only if x *and* y *are linearly dependent.*

Proof of Lemma 3.5: First let us discuss the case that x and y are linearly dependent. Then there exists a number $\alpha \in \mathbb{K}$ such that $x = \alpha\, y$ or $y = \alpha\, x$. Without restriction of generality we may assume that $x = \alpha\, y$, then

$$|\langle x, y \rangle| = |\langle \alpha\, y, y \rangle| = |\alpha \langle y, y \rangle| = |\alpha| \, \|y\|^2 = \sqrt{\alpha\, \overline{\alpha}\, \langle y, y \rangle}\, \|y\| = \sqrt{\langle \alpha\, y, \alpha\, y \rangle}\, \|y\| = \|x\| \, \|y\|.$$

Hence equality clearly holds for linearly dependent x and y.

If $x = \mathcal{O}$ or $y = \mathcal{O}$, then x and y are linearly independent; so this case is already covered, and we may from now on assume that $x \neq \mathcal{O}$ and $y \neq \mathcal{O}$.

If we have that $\langle x, y \rangle = 0$, then inequality is clearly true as $\|x\| \geq 0$ and $\|y\| \geq 0$, and hence $\langle x, y \rangle = 0 \leq \|x\| \, \|y\|$.

Now assume that $\langle x, y \rangle \neq 0$ and that x and y are linearly independent. Let α be a any complex number. Then $\alpha\, x + y \neq 0$ since x and y are linearly independent. From the non-degeneracy of the inner product (3.1.4)

$$0 < \|\alpha\, x + y\|^2 = \langle \alpha\, x + y, \alpha\, x + y \rangle = |\alpha|^2 \, \|x\|^2 + \big(\alpha\, \langle x, y \rangle + \overline{\alpha}\, \langle y, x \rangle\big) + \|y\|^2. \tag{3.1.17}$$

Notice that the expression in the brackets equals $2\,\Re(\alpha\, \langle x, y \rangle)$. Now we choose

$$\alpha = t\, |\langle x, y \rangle| \, \big(\langle x, y \rangle\big)^{-1}$$

with an arbitrary real number t. Then $|\alpha| = |t|$, and (3.1.17) implies

$$0 < t^2\,\|x\|^2 + 2\,t\,|\langle x, y\rangle| + \|y\|^2,$$

which is a quadratic function in t. The quadratic function of a real variable can be positive only if its discriminant is negative. Hence

$$4\,|\langle x, y\rangle|^2 - 4\,\|x\|^2\,\|y\|^2 < 0 \qquad \Leftrightarrow \qquad |\langle x, y\rangle|^2 < \|x\|^2\,\|y\|^2,$$

and taking the square-root give the desired inequality. □

The **triangle inequality** for the norm (3.1.15) follows from the Cauchy-Schwarz inequality, using the properties of the inner product: for all $x, y \in X$ we have

$$\begin{aligned}
\|x+y\|^2 &= \langle x+y, x+y\rangle \\
&= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle \\
&= \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} \\
&= \|x\|^2 + \|y\|^2 + 2\,\Re\langle x, y\rangle \\
&\leq \|x\|^2 + \|y\|^2 + 2|\Re\langle x, y\rangle| \\
&\leq \|x\|^2 + \|y\|^2 + 2|\langle x, y\rangle| \\
&\leq \|x\|^2 + 2\,\|x\|\,\|y\| + \|y\|^2 \\
&= (\|x\| + \|y\|)^2,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the second last step.

Lemma 3.6 (inner product space is a normed linear space)
Let X be an inner product space with the inner product $\langle\cdot,\cdot\rangle$. Then X is a normed linear space with the ***induced norm*** $\|x\| := \sqrt{\langle x, x\rangle}$.

The proof is left as an exercise.

Exercise 32 *Let X be a real inner product space with inner product $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$. Verify that $\|x\| := \sqrt{\langle x, x\rangle}$ defines a norm for X, thus making $(X, \|\cdot\|)$ a real normed linear space.*

Definition 3.7 (Hilbert space)
An inner product space (or pre-Hilbert space) X with inner product $\langle\cdot,\cdot\rangle$ is called a ***Hilbert space*** *if X with the norm $\|x\| := \sqrt{\langle x, x\rangle}$ is a Banach space, that is, if X with $\|x\| := \sqrt{\langle x, x\rangle}$ is a* ***complete*** *normed linear space. (Notation: Usually we use the letter H to denote a Hilbert space.)*

Example 3.8 (Hilbert spaces $\mathbb{R}^d$ and $\mathbb{C}^d$)
Both $\mathbb{R}^d$ and $\mathbb{C}^d$ with the inner product defined by (3.1.14) and (3.1.7), respectively, are Hilbert

spaces. The induced norm is the the usual Euclidean norm

$$\|\mathbf{x}\| = \left(\sum_{k=1}^{d} |x_k|^2 \right)^{1/2} \qquad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \text{ and } \mathbf{x}, \mathbf{y} \in \mathbb{C}^d, \text{ respectively.} \qquad \square$$

In fact, we have the more general situation that any finite dimensional inner product space is a Hilbert space.

Lemma 3.9 (finite dimensional inner product space is Hilbert space)
Any finite dimensional inner product space is a Hilbert space.

Exercise 33 *Give the proof of Lemma 3.9.*

The next lemma shows that the inner product is continuous.

Lemma 3.10 (inner product is continuous)
Let X be an inner product space with the inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ and with the induced norm $\|x\| := \sqrt{\langle x, x \rangle}$. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two convergent sequences in X with limits x and y, respectively, that is, $\lim_{n\to\infty} \|x_n - x\| = 0$ and $\lim_{n\to\infty} \|y_n - y\| = 0$. Then

$$\lim_{n\to\infty} \langle x_n, y_n \rangle = \langle x, y \rangle. \tag{3.1.18}$$

We note that in (3.1.18) the sequence $(\langle x_n, y_n \rangle)_{n\in\mathbb{N}}$ is just a sequence in $\mathbb{K}$ (that is, in $\mathbb{C}$ or $\mathbb{R}$).

Proof of Lemma 3.10: By (3.1.12) and (3.1.13)

$$\begin{aligned} \langle x_n, y_n \rangle - \langle x, y \rangle &= \langle x_n, y_n \rangle - \langle x, y_n \rangle + \langle x, y_n \rangle - \langle x, y \rangle \\ &= \langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle. \end{aligned} \tag{3.1.19}$$

The second term tends to zero due to the Cauchy-Schwarz inequality:

$$|\langle x, y_n - y \rangle| \leq \|x\| \, \|y_n - y\| \to 0 \qquad \text{as } n \to \infty. \tag{3.1.20}$$

To prove this for the first term, recall that the sequence $(y_n)_{n\in\mathbb{N}}$ is convergent and therefore bounded by Lemma 2.43 from Chapter 2. Therefore there exists a constant C such that $\|y_n\| \leq C$ for all $n \in \mathbb{N}$. Thus, from the Cauchy-Schwarz inequality

$$|\langle x_n - x, y_n \rangle| \leq \|x_n - x\| \, \|y_n\| \leq C \, \|x_n - x\| \to 0 \qquad \text{as } n \to \infty. \tag{3.1.21}$$

The required result follows now from (3.1.19), (3.1.20), and (3.1.21). Indeed

$$\big| \langle x_n, y_n \rangle - \langle x, y \rangle \big| \leq \big| \langle x_n - x, y_n \rangle \big| + \big| \langle x, y_n - y \rangle \big| \leq C \, \|x_n - x\| + \|x\| \, \|y_n - y\| \qquad \text{for all } n \in \mathbb{N},$$

and thus $\lim_{n\to\infty} |\langle x_n, y_n \rangle - \langle x, y \rangle| = 0$. $\square$

By direct calculation one checks the following two useful identities.

Lemma 3.11 (parallelogram identity and polarisation identity)
Let X be an inner product space with the inner product $\langle \cdot, \cdot \rangle$. For the norm $\|x\| := \sqrt{\langle x, x \rangle}$ induced by an inner product the following equalities hold:

(i) ***Parallelogram equality:***

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right) \qquad \text{for all } x, y \in X. \tag{3.1.22}$$

(ii) ***Polarisation identity:*** *If X is a complex inner product space, then*

$$4\langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2 + i\left(\|x+i\,y\|^2 - \|x-i\,y\|^2\right) \qquad \text{for all } x, y \in X. \tag{3.1.23}$$

Exercise 34 *Prove the parallelogram identity (3.1.22).*

Exercise 35 *Prove the polarisation identity (3.1.23).*

The inner product allows us to introduce the important concept of orthogonality.

Definition 3.12 (orthogonal vectors; vector orthogonal to a subset)
Let X be an inner product space with the inner product $\langle \cdot, \cdot \rangle$.

(i) Two elements $x, y \in X \setminus \{\mathcal{O}\}$ are ***orthogonal*** *if $\langle x, y \rangle = 0$. If $\langle x, y \rangle = 0$, we also write $x \perp y$.*

(ii) Let $M \subset X$ be a (finite or infinite) subset of X. We say that the ***vectors in M are orthogonal*** *if any two different $x, y \in M$ are orthogonal to each other. Then we also call M an* ***orthogonal set****.*

(iii) A vector $x \in X$ is said to be ***orthogonal to a subset*** *$M \subset X$ if x is orthogonal to every $y \in M$, that is, $\langle x, y \rangle = 0$ for all $y \in M$.*

Notice that two orthogonal non-zero vectors are linearly independent! In fact, if $M \subset X$ is a set of orthogonal vectors then the vectors in M are linearly independent. We will prove this later.

Notice also that for two **orthogonal vectors** x **and** y we have the **Pythagoras theorem**:

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2. \tag{3.1.24}$$

Exercise 36 *Prove Pythagoras theorem for an arbitrary inner product space X with inner product $\langle \cdot, \cdot \rangle$ and the corresponding induced norm $\|x\| = \sqrt{\langle x, x \rangle}$.*

Example 3.13 (orthogonal vectors in $\mathbb{R}^3$)
The Euclidean inner product of $\mathbb{R}^3$ is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle_2 := \sum_{k=1}^{3} x_k\, y_k = x_1\, y_1 + x_2\, y_2 + x_3\, y_3.$$

Here any two vectors are orthogonal (in the sense of Definition 3.12) if and only if they are perpendicular in the usual geometric sense. For example, the vectors $(1,2,1)^T$ and $(-1,1,-1)^T$ are orthogonal. □

Example 3.14 (orthogonal vectors in $\mathbb{R}^2$)
The Euclidean inner product for $\mathbb{R}^2$ is given by $\langle \mathbf{x}, \mathbf{y} \rangle_2 = x_1\, y_1 + x_2\, y_2$, where $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$. Any non-zero vector orthogonal to a given non-zero vector $\mathbf{x} = (x_1, x_2)^T$ is of the form $y = \alpha\,(-x_2, x_1)$, where $\alpha \in \mathbb{R} \setminus \{0\}$. □

Now we will discuss some more complicated examples of inner products and Hilbert spaces.

Example 3.15 (Hilbert space $\ell_2(\mathbb{N})$)
We have established in Chapter 2 that the sequence space $\ell_2(\mathbb{N})$ with the norm

$$\|x\|_2 = \left(\sum_{k\in\mathbb{N}} |x_k|^2 \right)^{1/2}, \qquad x = (x_k)_{k\in\mathbb{N}}, \tag{3.1.25}$$

is a Banach space. Now we define an inner product for this space by

$$\langle x, y \rangle_2 = \sum_{k\in\mathbb{N}} x_k\, \overline{y_k}. \tag{3.1.26}$$

Due to the Cauchy-Schwarz inequality in Lemma 3.5 (or Hölder's inequality in Lemma 2.14 with $p = q = 2$), this product is finite (and hence well defined); indeed, for $x, y \in \ell_2(\mathbb{N})$, we have $\|x\|_2 < \infty$ and $\|y\|_2 < \infty$ and thus

$$|\langle x, y \rangle_2| \le \|x\|_2\, \|y\|_2 < \infty.$$

The inner product (3.1.26) induces the norm (3.1.25), and hence $\ell_2(\mathbb{N})$ with the inner product (3.1.26) is an inner product space. As $\ell_2(\mathbb{N})$ is complete with respect to the norm (3.1.25), we know that $\ell_2(\mathbb{N})$ is a Hilbert space.

The following two elements are, for example, mutually orthogonal in ℓ_2:

$$x = (1, 0, 0, \ldots) \qquad \text{and} \qquad y = (0, y_2, y_3, \ldots),$$

indeed $\langle x, y \rangle_2 = 1 \cdot 0 + 0 \cdot y_2 + 0 \cdot y_3 + \ldots = 0$. □

Exercise 37 *Verify that (3.1.26) defines an inner product for $\ell_2(\mathbb{N})$.*

We summarise this as a corollary.

Corollary 3.16 ($\ell_2(\mathbb{N})$ is a Hilbert space)
The linear space $\ell_2(\mathbb{N})$ with the inner product

$$\langle x, y \rangle_2 := \sum_{k\in\mathbb{N}} x_k\, \overline{y_k}, \qquad x = (x_k)_{k\in\mathbb{N}},\ y = (y_k)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N}), \tag{3.1.27}$$

is a Hilbert space. The inner product (3.1.27) induces the 2-norm $\|\cdot\|_2$ for $\ell_2(\mathbb{N})$, defined by (2.2.16) with $p = 2$.

Example 3.17 (Hilbert space $L_2([a,b])$)
On the space $L_2([a,b])$ of square-integrable complex-valued functions (that is, all those measurable functions f on $[a,b]$ for which $\|f\|_{L_2([a,b])} = (\int_a^b |f(x)|^2 \, \mathrm{d}x)^{1/2} < \infty$) we define the inner product

$$\langle f, g \rangle_{L_2([a,b])} := \int_a^b f(x) \, \overline{g(x)} \, \mathrm{d}x. \tag{3.1.28}$$

As in the previous example, the Cauchy-Schwarz inequality shows that this number is finite if $f, g \in L_2([a,b])$. Furthermore, the inner product (3.1.28) induces the $L_2([a,b])$ norm

$$\|f\|_{L_2([a,b])} = \left(\int_a^b |f(x)|^2 \, \mathrm{d}x \right)^{1/2}$$

that was introduced in Definition 2.9. As we learnt that $L_2([a,b])$ with respect to this norm is complete, we know that $L_2([a,b])$ with the inner product (3.1.28) is a Hilbert space.

An example of orthogonal functions in $L_2([a,b])$ are $f(x) \equiv 1$ and any function g with the mean value zero (that is, $\int_a^b g(x) \, \mathrm{d}x = 0$). Indeed

$$\langle f, g \rangle_{L_2([a,b])} = \int_a^b f(x) \, \overline{g(x)} \, \mathrm{d}x = \int_a^b \overline{g(x)} \, \mathrm{d}x = \overline{\int_a^b g(x) \, \mathrm{d}x} = 0.$$

For example, if $a = -\pi, b = \pi$, then $g(x) = \sin x$ is orthogonal to $f(x) \equiv 1$. □

Corollary 3.18 ($L_2([a,b])$ and $L_2(\mathbb{R})$ are Hilbert spaces)

(i) The space $L_2([a,b])$ with the inner product

$$\langle f, g \rangle_{L_2([a,b])} := \int_a^b f(x) \, \overline{g(x)} \, \mathrm{d}x$$

is a Hilbert space. This inner product induces the norm $\| \cdot \|_{L_2([a,b])}$, defined by (2.2.20) with $p = 2$.

(ii) The space $L_2(\mathbb{R})$ with the inner product

$$\langle f, g \rangle_{L_2(\mathbb{R})} := \int_{\mathbb{R}} f(x) \, \overline{g(x)} \, \mathrm{d}x$$

is a Hilbert space. This inner product induces the norm $\| \cdot \|_{L_2(\mathbb{R})}$, defined by (2.2.21) with $p = 2$.

The function spaces $L_2([a,b])$ and $L_2(\mathbb{R})$ will play an important role in this course. Therefore we discuss some more examples of (sets of) orthogonal functions in $L_2([a,b])$ for specific choices of $[a,b]$.

Example 3.19 (complex trigonometric polynomials)
The 2π-periodic complex trigonometric functions

$$e_k(x) = \alpha_k \, e^{ikx}, \qquad k \in \mathbb{Z}, \tag{3.1.29}$$

with normalisation factors $\alpha_k \in \mathbb{C}$, are in the space $L_2([-\pi,\pi])$. Indeed,

$$\|e_k\|_{L_2([-\pi,\pi])} = \left(\int_{-\pi}^{\pi} |e_k(x)|^2 \,\mathrm{d}x\right)^{1/2} = \left(|\alpha_k|^2 \int_{-\pi}^{\pi} |e^{ikx}|^2 \,\mathrm{d}x\right)^{1/2} = |\alpha_k| \left(\int_{-\pi}^{\pi} 1 \,\mathrm{d}x\right)^{1/2} = |\alpha_k|\,\sqrt{2\pi}.$$

One easily checks that these functions are mutually orthogonal:

$$\int_{-\pi}^{\pi} e_k(x)\,\overline{e_m(x)}\,\mathrm{d}x = \alpha_k\,\overline{\alpha_m} \int_{-\pi}^{\pi} e^{i(k-m)x}\,\mathrm{d}x = \begin{cases} 2\pi\,|\alpha_k|^2 & \text{if } k=m, \\ \left[\alpha_k\,\overline{\alpha_m}\,(i(k-m))^{-1}\,e^{i(k-m)x}\right]_{-\pi}^{\pi} = 0 & \text{if } k \neq m, \end{cases}$$

where we have used that $e^{-in\pi} = e^{in\pi}$ for all $n \in \mathbb{N}$, due to the 2π-periodicity of e^{-inx}, $n \in \mathbb{Z}$. The functions e_k, $k \in \mathbb{Z}$, and any linear combinations of these are called **complex trigonometric polynomials**. We will say that e_k and e_{-k} are the **complex trigonometric basis polynomials of degree** k, and we will refer to the set $\{e_k \,:\, k \in \mathbb{Z}\}$ as the **(set of) complex trigonometric basis polynomials**. The set span $\{e_k \,:\, k = -n,\ldots,n\}$ is the **space of complex trigonometric polynomials of degree** $\leq n$. □

You may have encountered 2π-periodic **(real) trigonometric basis polynomials** in previous courses as the set of functions $1, \cos x, \sin x, \cos(2x), \sin(2x), \ldots, \cos(kx), \sin(kx), \ldots$. These are of course related to the functions e^{ikx} via Euler's formula

$$e^{ikx} = \cos(kx) + i\,\sin(kx) \qquad \text{and} \qquad e^{-ikx} = \cos(kx) - i\,\sin(kx),$$

from which we see that

$$\cos(kx) = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right) \qquad \text{and} \qquad \sin(kx) = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right).$$

As we consider complex-valued functions in this course, it is convenient for us to use the complex trigonometric basis functions rather then then real trigonometric basis functions.

Example 3.20 (characteristic functions)
Denote by $\chi_{\mathcal{I}}$ the **characteristic function** of the interval $\mathcal{I}$, that is

$$\chi_{\mathcal{I}}(x) := \begin{cases} 1 & \text{if } x \in \mathcal{I}, \\ 0 & \text{if } x \notin \mathcal{I}. \end{cases}$$

Let $\mathcal{I}, \mathcal{J} \subset \mathbb{R}$ be two bounded intervals such that $\mathcal{I} \cap \mathcal{J} = \emptyset$. Then the functions $\chi_{\mathcal{I}}$ and $\chi_{\mathcal{J}}$ are orthogonal with respect to the inner product

$$\langle f, g \rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,\mathrm{d}x.$$

This is clear since $\chi_{\mathcal{I}}(x)\,\chi_{\mathcal{J}}(x) = 0$ for all $x \in \mathbb{R}$. – In particular, if $\mathcal{I}_k = [k, k+1)$, then the functions $\chi_{\mathcal{I}_k}$ and $\chi_{\mathcal{I}_m}$ are orthogonal for distinct $k, m \in \mathbb{Z}$. In fact, this statement is still true if we choose $\mathcal{I}_k = [k, k+1]$, so that distinct intervals may have a common boundary point. □

The next example will be very important later-on and will in fact furnish our standard example of a wavelet.

Example 3.21 (Haar scaling function and Haar wavelet)
Let $\phi(x) := \chi_{[0,1)}(x)$ be the characteristic function of the half-open unit interval $[0, 1)$. This function is in the context of wavelets often referred to as the **Haar scaling function**. Define

$$\psi(x) := \phi(2x) - \phi(2x - 1).$$

This function is called the **Haar wavelet**. From its definition, we see that

$$\psi(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2), \\ -1 & \text{if } x \in [1/2, 1), \\ 0 & \text{if } x \in (-\infty, 0) \cup [1, \infty). \end{cases}$$

It is easy to verify that the Haar scaling function and the Haar wavelet are orthogonal in $L_2(\mathbb{R})$, that is,

$$\langle \phi, \psi \rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} \phi(x)\,\overline{\psi(x)}\,\mathrm{d}x = 0. \tag{3.1.30}$$

The proof of the orthogonality (3.1.30) is left as an exercise. □

Exercise 38 *Verify the orthogonality (3.1.30) of the Haar scaling function and the Haar wavelet.*

Remark 3.22 It is not possible to define an inner product for the spaces $\ell_p(\mathbb{N})$ or $L_p([a, b])$ or $L_p(\mathbb{R})$ with $p \neq 2$.

Similarly to the case of normed linear spaces, we introduce subspaces of an inner product space.

Definition 3.23 (subspace of an inner product space)
*Let X be an inner product space with the inner product $\langle \cdot, \cdot \rangle$. A **subspace** Y of X is a subspace of the linear space X endowed with the inner product $\langle \cdot, \cdot \rangle$ restricted to Y.*

The previous definition makes it clear that a subspace Y of a Hilbert space H is an inner product space. The fact that a Hilbert space is a complete inner product space does in general **not** imply that a subspace is also complete! Instead we the following statement.

Lemma 3.24 (closed subspaces of a Hilbert space are complete)
*Let H be a Hilbert space. A subspace Y of H is complete (and hence a Hilbert space) if Y is a **closed subspace** of H.*

Since a Hilbert space H is a linear space with additional properties, it has a **dimension**, defined as the dimension of the linear space H. From Lemma 3.24 we draw the following conclusion.

Lemma 3.25 (finite dimensional subspaces of a Hilbert space are complete)
Let H be a Hilbert space and let Y be a finite dimensional subspace of H. Then Y is closed and complete and hence also a Hilbert space.

The proofs of Lemma 3.24 and Lemma 3.25 are left as exercises.

Exercise 39 *Prove Lemma 3.24.*

Exercise 40 *Prove Lemma 3.25.*

3.2 Best Approximation in Hilbert Spaces

In this section, we discuss the concept of **distance** and **best approximation in a subset**. For a closed subspace Y of a Hilbert space H, the concept of best approximation in Y will lead us to the notion of an **orthogonal projection onto** Y.

Definition 3.26 (distance in a Hilbert space)
Let X be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|x\| := \sqrt{\langle x, x \rangle}$. The distance $\mathrm{dist}(x,y)$ of x and y in X is measured with the norm via

$$\mathrm{dist}(x,y) := \|x-y\|.$$

Definition 3.27 (distance from a subset and best approximation in a subset)
*Let X be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|x\| := \sqrt{\langle x, x \rangle}$, and let $M \subset X$ be a subset of X. For any (fixed) $x \in X$, the **distance from** x **to** M is defined as*

$$\mathrm{dist}(x,M) := \inf_{y \in M} \|x-y\|. \qquad (3.2.1)$$

An element $x^* \in M$, where the infimum in (3.2.1) is attained, that is, which satisfies

$$\|x - x^*\| = \mathrm{dist}(x,M),$$

*is called a **best approximation of** x **in** M.*

We will now address the questions, whether a best approximation exists and, if it exists, whether it is uniquely determined.

Definition 3.28 (convex subset)
*Let X be a linear space, and let $M \subset X$. The subset M is called **convex**, if for every $x, y \in M$ the vectors*

$$z = y + \alpha\,(x-y) = \alpha\, x + (1-\alpha)\, y, \qquad \alpha \in (0,1), \qquad (3.2.2)$$

also belong to M. (Note that the set (3.2.2) is just the 'straight line' connecting x and y, excluding the end points.)

We note that for sets in $\mathbb{R}^2$ and $\mathbb{R}^3$ this definition describes just convexity in the usual geometric sense.

Example 3.29 (convex sets)

(a) The closed unit ball $\widetilde{B}(0;1)$ in $\mathbb{R}^3$ is convex.

(b) The unit sphere $S(0;1)$ in $\mathbb{R}^3$ is not convex.

(c) The set of constant functions in $C(\mathbb{R})$ is convex. □

Lemma 3.30 (subspaces are convex)
Any subspace Y of a linear space X is convex.

Proof of Lemma 3.30: Let $x, y \in Y$ be two arbitrary vectors. For any $\alpha \in (0,1)$ the vector

$$z = y + \alpha\,(x-y) = \alpha\,x + (1-\alpha)\,y$$

is a linear combination of x and y and lies also in the subspace Y. Hence Y is convex. □

In a convex closed subset of a Hilbert space there exists a best approximation.

Theorem 3.31 (best approximation exists in closed convex subset)
Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|x\| := \sqrt{\langle x,x\rangle}$. Let $M \subset H$ be a ***closed convex*** *subset of H. Then, for every $x \in H$ there exists a* ***unique*** *$x^* \in M$ such that*

$$\mathrm{dist}(x,M) = \|x - x^*\|,$$

that is, there exists a ***unique best approximation*** *of x in M.*

Proof of Theorem 3.31: We first show the existence of a best approximation. Then we show that it is unique.

Existence: By definition of an infimum, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ in M such that

$$d_n := \|x - y_n\| \to \mathrm{dist}(x,M) =: d, \qquad \text{as } n \to \infty. \tag{3.2.3}$$

We now show that this sequence is a Cauchy sequence: Due to the convexity of M, for any $m, n \in \mathbb{N}$ the element $(y_n + y_m)/2$ is also in M. Thus

$$\|(y_n + y_m)/2 - x\| \geq \mathrm{dist}(x,M) = d. \tag{3.2.4}$$

Thus by the parallelogram equality (3.1.22), we have

$$\begin{aligned} \|y_n + y_m - 2\,x\|^2 + \|y_n - y_m\|^2 &= \|(y_n - x) + (y_m - x)\|^2 + \|(y_n - x) - (y_m - x)\|^2 \\ &= 2\,\left(\|y_n - x\|^2 + \|y_m - x\|^2\right), \end{aligned}$$

which can be rearranged to yield

$$\begin{aligned} \|y_n - y_m\|^2 &= -\,\|y_n + y_m - 2\,x\|^2 + 2\,\left(\|y_n - x\|^2 + \|y_m - x\|^2\right) \\ &= -\,4\,\|(y_n + y_m)/2 - x\|^2 + 2\,\left(\|y_n - x\|^2 + \|y_m - x\|^2\right) \\ &\leq -\,4\,d^2 + 2\,(d_n^2 + d_m^2), \end{aligned} \tag{3.2.5}$$

where we have used (3.2.4) in the last step. The right-hand side tends to zero as m and n tend to infinity, due to (3.2.3). Thus for any $\epsilon > 0$, there exists an $N = N(\epsilon) \in \mathbb{N}$ such that

$$\|y_n - y_m\| \leq \sqrt{2\,(d_n^2 + d_m^2) - 4\,d^2} < \epsilon \qquad \text{for all } n, m \geq N,$$

and we see $(y_n)_{n\in\mathbb{N}}$ is indeed a Cauchy sequence in H.

Since the Hilbert space H is complete, we know that this Cauchy sequence $(y_n)_{n\in\mathbb{N}}$ converges to a limit $x^* \in H$, that is, $x^* := \lim_{n\to\infty} y_n$ and $x^* \in H$. Since $y_n \in M$ for all $n \in \mathbb{N}$, the limit x^* is an accumulation point of M. Since M is closed the accumulation point x^* lies also in M. Thus we see that $(y_n)_{n\in\mathbb{N}}$ converges to a limit $x^* \in M$.

Moreover, $\|x^* - x\| \geq d$ as x^* lies in M, and also from the triangle inequality

$$d \leq \|x - x^*\| \leq \|x - y_n\| + \|y_n - x^*\| \to d \qquad \text{as } n \to \infty.$$

From the sandwich theorem, this implies that $\|x - x^*\| = d$, and hence x^* is a best approximation of x.

Uniqueness: Suppose that there are two distinct vectors $x^* \in M$ and $z \in M$ that are best approximations, that is

$$\|x - x^*\| = \|x - z\| = \mathrm{dist}(x, M) = d.$$

Then all vectors of the form $\alpha\, x^* + (1-\alpha)\, z$, where $\alpha \in (0,1)$, lie in M, as M is convex. Also all such vectors satisfy (using the triangle inequality)

$$\begin{aligned} d &\leq \|\alpha\, x^* + (1-\alpha)\, z - x\| \\ &= \|\alpha\,(x^* - x) + (1-\alpha)\,(z - x)\| \\ &\leq \alpha\,\|x^* - x\| + (1-\alpha)\,\|z - x\| \\ &= \alpha\, d + (1-\alpha)\, d \;=\; d, \end{aligned}$$

which verifies that

$$\|\alpha\, x^* + (1-\alpha)\, z - x\| = d \qquad \text{for all } \alpha \in [0,1]. \tag{3.2.6}$$

Now using the parallelogram equality (3.1.22) again (replace in the first line of (3.2.5) y_n and y_m by x^* and z, respectively), we see that

$$\begin{aligned} \|x^* - z\|^2 &= -\,\|x^* + z - 2x\|^2 + 2\left(\|x^* - x\|^2 + \|z - x\|^2\right) \\ &= -4\,\|(x^* + z)/2 - x\|^2 + 2\left(\|x^* - x\|^2 + \|z - x\|^2\right) \\ &= -4\,\delta^2 + 2\,(\delta^2 + \delta^2) \;=\; 0, \end{aligned}$$

where we have used (3.2.6) with $\alpha = 1/2$ in the second-last step. Therefore $x^* = z$, and we have verified that the best approximation is unique. □

We come now to the main characterisation of the best approximation in the case of a subspace.

Theorem 3.32 (best approx. $x^* \in Y$ of x satisfies $(x - x^*) \perp Y$ if Y closed subspace) *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|x\| := \sqrt{\langle x, x\rangle}$. Suppose Y is a* ***closed subspace*** *of H. Then $x^* \in Y$ is the unique best approximation of $x \in H$ in Y, if and only if $x - x^*$* ***is orthogonal to*** *Y.*

Before we prove the theorem, we remark that, from Theorem 3.31, it is clear that the best approximation exists and is unique, since the closed subspace is both closed and convex.

Proof of Theorem 3.32: $\Rightarrow$: Let x^* be the unique best approximation of $x \in H$, and consider an arbitrary vector $y \in Y$ and let $\alpha := \langle x - x^*, y\rangle$. Without restriction, we may assume that $\|y\| = 1$. As y and x^* are in Y and as Y is subspace, the vector $x^* + \alpha\, y$ is also in Y. Then

$$\begin{aligned} \|x - (x^* + \alpha\, y)\|^2 &= \big\langle (x - x^*) - \alpha\, y, (x - x^*) - \alpha\, y \big\rangle \\ &= \|x - x^*\|^2 - \overline{\alpha}\, \langle x - x^*, y\rangle - \alpha\, \langle y, x - x^*\rangle + |\alpha|^2\, \|y\|^2 \\ &= \|x - x^*\|^2 - \overline{\alpha}\, \alpha - \alpha\, \overline{\alpha} + |\alpha|^2 < 0 \\ &= \|x - x^*\|^2 - |\alpha|^2, \end{aligned}$$

where we have used the definition of α and $\|y\| = 1$. As x^* is the unique best approximation of x in Y and as $x^* + \alpha\, y \in Y$, we also know from the previous estimate that

$$\|x - x^*\|^2 \leq \|x - (x^* + \alpha\, y)\|^2 = \|x - x^*\|^2 - |\alpha|^2$$

This can only hold true if $\alpha = 0$, yielding from the definition of α that $\langle x - x^*, y\rangle = 0$. As $y \in Y$ was arbitrary, we have shown that $x - x^*$ is indeed orthogonal to Y.

$\Leftarrow$: Let $x^* \in Y$ and $(x - x^*) \perp Y$, that is $\langle x - x^*, y\rangle = 0$ for all $y \in Y$. This particularly means that for any $y \in Y$ also $\langle x - x^*, x^* - y\rangle = 0$ (since $x^* - y \in Y$ as $x^*, y \in Y$ and as Y is a linear space). Hence, by Pythagoras theorem (3.1.24), we have (using $\langle x - x^*, x^* - y\rangle = 0$)

$$\|x - y\|^2 = \|(x - x^*) + (x^* - y)\|^2 = \big\langle (x - x^*) + (x^* - y), (x - x^*) + (x^* - y) \big\rangle = \|x - x^*\|^2 + \|x^* - y\|^2$$

for all $y \in Y$. As $\|x^* - y\|^2 \geq 0$, this implies

$$\|x - y\| \geq \|x - x^*\| \qquad \text{for all } y \in Y,$$

with equality only if $y = x^*$. Thus x^* is indeed the unique best approximation of x in Y. □

Let us have a look at the special situation of a **finite dimensional subspace** Y. Since a finite dimensional subspace of a Hilbert space is always **convex and closed**, any $x \in H$ has a unique best approximation in $x^* \in Y$ (due Theorem 3.31). Assume that Y is an n-dimensional subspace and that $\{\psi_1, \psi_2, \ldots, \psi_n\}$ is a basis for Y. Then, given $x \in H$, we can write its best approximation x^* in Y as

$$x^* = \sum_{j=1}^{n} \alpha_j\, \psi_j \tag{3.2.7}$$

with coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$, which we have to determine. From Theorem 3.32,

$$\langle x - x^*, y\rangle = 0 \qquad \text{for all } y \in Y.$$

Since $\{\psi_1, \psi_2, \ldots, \psi_n\}$ is a basis of Y, this is equivalent to

$$\langle x - x^*, \psi_k \rangle = 0 \quad \text{for all } k = 1, 2, \ldots, n \qquad \Leftrightarrow \qquad \langle x, \psi_k \rangle = \langle x^*, \psi_k \rangle \quad \text{for all } k = 1, 2, \ldots, n.$$

Hence, inserting the representation (3.2.7) of x^*, we have

$$\langle x, \psi_k \rangle = \langle x^*, \psi_k \rangle = \left\langle \sum_{j=1}^{n} \alpha_j \, \psi_j, \psi_k \right\rangle = \sum_{j=1}^{n} \alpha_j \, \langle \psi_j, \psi_k \rangle, \qquad k = 1, 2, \ldots, n.$$

This defines a linear system for the coefficient vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$, namely

$$\mathbf{A} \, \boldsymbol{\alpha} = \mathbf{b},$$

with the Hermitian matrix $\mathbf{A} = [\langle \psi_j, \psi_k \rangle]_{j,k=1,2,\ldots,n}$, and the right-hand side $\mathbf{b} = (\langle x, \psi_k \rangle)^T_{k=1,2,\ldots,n}$.

This leads us to the following corollary.

Corollary 3.33 (form of best approximation in a finite dimensional subspace) *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|x\| := \sqrt{\langle x, x \rangle}$, and let Y be a finite dimensional subspace of H. Suppose $\{\psi_1, \psi_2, \ldots, \psi_n\}$ forms a basis for the subspace Y. Then the* ***best approximation*** *$x^* \in Y$ of $x \in H$ in Y has the unique representation*

$$x^* = \sum_{j=1}^{n} \alpha_j \, \psi_j,$$

where the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$ are the ***unique*** *solution of the following* ***linear system****:*

$$\langle x, \psi_k \rangle = \sum_{j=1}^{n} \alpha_j \, \langle \psi_j, \psi_k \rangle, \qquad k = 1, 2, \ldots, n. \tag{3.2.8}$$

Proof of Corollary 3.33: From the considerations before the corollary it remains only to show that the linear system (3.2.8) is uniquely solvable. We saw already that we can rewrite (3.2.8) as $\mathbf{A} \, \boldsymbol{\alpha} = (\langle x, \psi_k \rangle)^T_{k=1,2,\ldots,n}$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ and the unitary matrix $\mathbf{A} = [\langle \psi_j, \psi_k \rangle]_{j,k=1,2,\ldots,n}$. From

$$\boldsymbol{\alpha}^T \mathbf{A} \, \overline{\boldsymbol{\alpha}} = \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \, \overline{\alpha_k} \, \langle \psi_j, \psi_k \rangle = \left\langle \sum_{j=1}^{n} \alpha_j \, \psi_j, \sum_{k=1}^{n} \alpha_k \, \psi_k \right\rangle = \left\| \sum_{j=1}^{n} \alpha_j \, \psi_j \right\| \geq 0,$$

we see that the matrix $\mathbf{A}$ is positive semi-definite. Moreover, we have $\boldsymbol{\alpha}^T \mathbf{A} \, \overline{\boldsymbol{\alpha}} = 0$ if and only if $\sum_{j=1}^{n} \alpha_j \, \psi_j = 0$ which implies $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T = (0, 0, \ldots, 0)^T$ (as $\psi_1, \psi_2, \ldots, \psi_n$ are linearly independent). Hence we see that $\boldsymbol{\alpha}^T \mathbf{A} \, \overline{\boldsymbol{\alpha}} \geq 0$ with equality only if $\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}} = \mathbf{0}$, showing that $\mathbf{A}$ is positive definite and hence in particular invertible. (Indeed, if $\mathbf{A} \, \overline{\boldsymbol{\alpha}} = \mathbf{0}$, then we have $\boldsymbol{\alpha}^T \mathbf{A} \, \overline{\boldsymbol{\alpha}} = 0$, and we know that this is only true for $\boldsymbol{\alpha} = \mathbf{0}$. Thus $\mathbf{A} \, \overline{\boldsymbol{\alpha}} = \mathbf{0}$ implies $\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}} = \mathbf{0}$, and hence $\mathbf{A}$ is invertible.) Thus the linear system (3.2.8) has indeed a unique solution. $\square$

> **Definition 3.34 (Gram matrix/Gramian matrix)**
> *Let H be a Hilbert space with inner product $\langle \cdot,\cdot\rangle$, and let Y be an n-dimensional subspace of Y. Let $\psi_1,\psi_2,\ldots,\psi_n$ be a basis of Y. The matrix $\mathbf{A} = [\langle \psi_j,\psi_k\rangle]_{j,k=1,2,\ldots,n}$ is called the* ***Gram matrix*** *(or* ***Gramian Matrix****) of the basis $\psi_1,\psi_2,\ldots,\psi_n$.*

From the definition of the Gram matrix $\mathbf{A}$ it is clear that $\mathbf{A}$ is **Hermitian** (that is, $\overline{\mathbf{A}}^T = \mathbf{A}$). We saw in the proof of Corollary 3.33 that any Gram matrix is **positive definite** and **invertible**. (A square matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is called positive definite if $\mathbf{A}$ is Hermitian and if for all $\boldsymbol{\alpha} \in \mathbb{C}^n$ we have $\overline{\boldsymbol{\alpha}}^T \mathbf{A}\,\boldsymbol{\alpha} \geq 0$ with equality only if $\boldsymbol{\alpha} = \mathbf{0}$.)

Exercise 41 *Let X be an inner product space with inner product $\langle \cdot,\cdot\rangle$, and let $Y \subset X$ be a finite dimensional subspace. Show that the subspace Y is closed.*

We want now to compute the best approximation in two examples:

Example 3.35 (best approximation in $\mathbb{R}^3$)
Consider $\mathbb{R}^3$ with the Euclidean inner product

$$\langle \mathbf{x},\mathbf{y}\rangle_2 = \sum_{j=1}^{3} x_j\, y_j = x_1\, y_1 + x_2\, y_2 + x_3\, y_3$$

and the induced Euclidean norm $\|\mathbf{x}\|_2 = \sqrt{\sum_{j=1}^3 |x_j|^2}$. Let Y be the linear subspace

$$Y = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\},$$

and let $\mathbf{x} = (1,0,3)^T$. We want to find the best approximation $\mathbf{x}^*$ of $\mathbf{x}$ in Y with the help of Corollary 3.33.

Solution: First we use common sense to guess the answer: We observe that Y is just the hyperplane $z = 0$ in $\mathbb{R}^3$. Hence we expect that the best approximation of $\mathbf{x} = (1,0,3)^T$ in Y is given by $\mathbf{x}^* = (1,0,0)^T$ and that $\operatorname{dist}(\mathbf{x},Y) = 3$. We will now verify this by a proper computation.

Setting

$$\mathbf{x}^* = \alpha_1 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

the Gram matrix is given by

$$\mathbf{A} = \begin{pmatrix} \big\langle (2,1,0)^T,(2,1,0)^T \big\rangle_2 & \big\langle (-1,1,0)^T,(2,1,0)^T \big\rangle_2 \\ \big\langle (2,1,0)^T,(-1,1,0)^T \big\rangle_2 & \big\langle (-1,1,0)^T,(-1,1,0)^T \big\rangle_2 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$$

and

$$\big\langle (1,0,3)^T,(2,1,0)^T \big\rangle_2 = 2 \qquad \text{and} \qquad \big\langle (1,0,3)^T,(-1,1,0)^T \big\rangle_2 = -1.$$

Thus the linear system is

$$\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Solving the linear system we find

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix},$$

and the best approximation $\mathbf{x}^*$ is given by

$$\mathbf{x}^* = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

as expected. Furthermore, $\mathrm{dist}(\mathbf{x}, Y) = \|\mathbf{x} - \mathbf{x}^*\|_2 = \|(0,0,3)^T\|_2 = 3$ as expected. □

Example 3.36 (best approximation by polynomials in $L_2([0,1])$)
Let $H = L_2([0,1])$ be the space of real-valued square-integrable functions, endowed with the inner product

$$\langle f, g \rangle_{L_2([0,1])} := \int_0^1 f(t)\, \overline{g(t)}\, \mathrm{d}t$$

and the induced norm

$$\|f\|_{L_2([0,1])} := \sqrt{\langle f, f \rangle_{L_2([0,1])}} = \left(\int_0^1 |f(t)|^2 \, \mathrm{d}t \right)^{1/2}.$$

Let $Y = \mathrm{span}\,\{1, t, t^2\} = \Pi_2([0,1])$ be the space of polynomials of degree ≤ 2 on $[0,1]$ with real coefficients. We want to compute the best approximation of $f(t) = e^t$ in Y with the help of Corollary 3.33. First we compute the Gram matrix (where we write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{L_2([0,1])}$ for brevity):

$$\mathbf{A} = \begin{pmatrix} \langle 1,1 \rangle & \langle 1,t \rangle & \langle 1,t^2 \rangle \\ \langle t,1 \rangle & \langle t,t \rangle & \langle t,t^2 \rangle \\ \langle t^2,1 \rangle & \langle t^2,t \rangle & \langle t^2,t^2 \rangle \end{pmatrix} = \begin{pmatrix} \int_0^1 1\,\mathrm{d}t & \int_0^1 t\,\mathrm{d}t & \int_0^1 t^2\,\mathrm{d}t \\ \int_0^1 t\,\mathrm{d}t & \int_0^1 t^2\,\mathrm{d}t & \int_0^1 t^3\,\mathrm{d}t \\ \int_0^1 t^2\,\mathrm{d}t & \int_0^1 t^3\,\mathrm{d}t & \int_0^1 t^4\,\mathrm{d}t \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}.$$

Next we compute the right-hand side

$$\mathbf{b} = \begin{pmatrix} \langle e^t, 1 \rangle \\ \langle e^t, t \rangle \\ \langle e^t, t^2 \rangle \end{pmatrix} = \begin{pmatrix} \int_0^1 e^t\,\mathrm{d}t \\ \int_0^1 e^t\, t\,\mathrm{d}t \\ \int_0^1 e^t\, t^2\,\mathrm{d}t \end{pmatrix} = \begin{pmatrix} e-1 \\ 1 \\ e-2 \end{pmatrix}.$$

The coefficients $\alpha_1, \alpha_2, \alpha_3$ in best approximation $f^*(t) = \alpha_1 + \alpha_2\, t + \alpha_2\, t^2$ are then the solutions of the linear system $\mathbf{A}\,\boldsymbol{\alpha} = \mathbf{b}$, given by

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} e-1 \\ 1 \\ e-2 \end{pmatrix}.$$

Thus we have, computing the inverse matrix $\mathbf{A}^{-1}$,

$$\boldsymbol{\alpha} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix} \begin{pmatrix} e-1 \\ 1 \\ e-2 \end{pmatrix} = \begin{pmatrix} -105+39\,e \\ 588-216\,e \\ -570+210\,e \end{pmatrix} \approx \begin{pmatrix} 1.0130 \\ 0.8511 \\ 0.8392 \end{pmatrix},$$

and hence the best approximation of $f(t) = e^t$ in Y is given by the polynomial

$$f^*(t) = 1.0130 + 0.8511\,t + 0.8392\,t^2. \qquad \square$$

Exercise 42 *Let $\mathbb{R}^3$ be the Euclidean space with the Euclidean inner product $\langle \mathbf{x}, \mathbf{y} \rangle_2 = \sum_{j=1}^3 x_j\, y_j$. Using Corollary 3.33, compute the best approximation of the vector $\mathbf{x} = (1, 4, 2)^T$ in the subspace*

$$Y = \text{span}\,\left\{(2,0,1)^T, (1,0,-1)^T\right\}.$$

Definition 3.37 (direct sum and orthogonal direct sum)

*(i) A linear space X is said to be the **direct sum** of two subspaces Y and Z, written as*

$$X = Y \oplus Z,$$

*if every $x \in X$ has a **unique** representation*

$$x = y + z \qquad \text{with } y \in Y \text{ and } z \in Z.$$

*The subspaces Y and Z are called a **complementary pair** of subspaces of X.*

*(ii) If X is an inner product space (with inner product $\langle \cdot, \cdot \rangle$) and Y and Z are two subspaces such that $X = Y \oplus Z$ and $Y \perp Z$ (that is, $\langle y, z \rangle = 0$ for any $y \in Y$ and $z \in Z$), then the direct sum $X = Y \oplus Z$ is called an **orthogonal sum**.*

We will now prove that every Hilbert space can be represented as a direct sum of a **closed subspace** and its **orthogonal complement**.

Definition 3.38 (orthogonal complement of a subset)
*Let X be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and let $M \subset X$ be a subset. Then, the **orthogonal complement of** M (or the **annihilator of** M) is defined to be the set*

$$M^\perp := \{x \in X \,:\, x \perp M\} = \{x \in X \,:\, \langle x, y \rangle = 0 \text{ for all } y \in M\}.$$

The orthogonal complement is always a closed subspace.

Lemma 3.39 (orthogonal complement in a Hilbert space is a closed subspace)
*Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $M \subset H$ be a subset. The orthogonal complement $M^\perp$ of M is a **closed subspace** of H.*

Proof of Lemma 3.39: First of all, if $x_1, x_2 \in M^\perp$ and $\alpha_1, \alpha_2 \in \mathbb{K}$, then the linearity of the inner product in the first argument yields

$$\langle \alpha_1 x_1 + \alpha_2 x_2, y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2, y \rangle = 0 \qquad \text{for all } y \in M.$$

Hence $M^\perp$ is closed under vector addition and scalar multiplication and is hence a subspace.

Next, take a Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset M^\perp$. Since H is a Hilbert space, the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ converges in H to a unique limit $x \in H$. Then, the continuity of the inner product (see Lemma 3.10) allows us to conclude that

$$\langle x, y \rangle = \lim_{n\to\infty} \underbrace{\langle x_n, y \rangle}_{=0} = 0 \qquad \text{for all } y \in M.$$

This means that the limit x is also in $M^\perp$ and hence that $M^\perp$ is closed. □

Theorem 3.40 (Hilbert space as orthogonal sum of closed subspace Y and $Y^\perp$)
Let Y be a closed subspace of the Hilbert space H. Then $H = Y \oplus Y^\perp$ and the direct sum is an orthogonal sum.

Proof of Theorem 3.40: By Theorem 3.31 and Theorem 3.32 for any $x \in H$ there exist a unique best approximation $x^* \in Y$ such that $\mathrm{dist}(x, Y) = \|x - x^*\|$ and $(x - x^*) \perp Y$. Hence, if we define $z := x - x^*$ then we have obviously

$$x = x^* + (x - x^*) = x^* + z$$

and $x^* \in Y$ and $z = x - x^* \in Y^\perp$. Hence any $x \in H$ can be written as a sum of an element in Y and an element in $Y^\perp$. By definition of the orthogonal complement $Y^\perp$ is orthogonal to Y. It remains to show that this representation $x = x^* + z$ with the best approximation $x^* \in Y$ and $z = x - x^* \in Y^\perp$ is unique.

Suppose that we have two representations

$$x = x_1 + z_1 = x_2 + z_2$$

with $x_1, x_2 \in Y$ and $z_1, z_1 \in Y^\perp$. Then,

$$x_1 - x_2 = z_2 - z_1$$

and taking the inner product with $x_1 - x_2$ leads to

$$\|x_1 - x_2\|^2 = \langle x_1 - x_2, z_2 - z_1 \rangle = 0,$$

where we have used that $x_1 - x_2 \in Y$ and $z_2 - z_1 \in Y^\perp$. Thus, we have $x_1 = x_2$ and then also $z_1 = z_2$. □

We compute the orthogonal complement for some examples.

Example 3.41 (orthogonal complement in $\mathbb{R}^3$)
Let $\mathbb{R}^3$ be the usual Euclidean space with the Euclidean inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{3} x_j\, y_j$. Let Y be the subspace

$$Y := \left\{ t\,(1,2,3)^T \,:\, t \in \mathbb{R} \right\}.$$

Find the orthogonal complement of Y.

Solution: Since $\dim Y = 1$ and $\dim \mathbb{R}^3 = 3$, we know that $Y^\perp$ is a 2-dimensional subspace, that is, a plain through the origin. This plane will be spanned by any two linearly independent vectors that are orthogonal to $(1,2,3)^T$. For example, we can choose $(1,1,-1)^T$ and $(-3,0,1)^T$. Thus $Y^\perp$ is the linear space

$$Y^\perp = \left\{ \alpha\,(1,1,-1)^T + \beta\,(-3,0,1)^T \,:\, \alpha, \beta \in \mathbb{R} \right\}.$$

The Hilbert space $\mathbb{R}^3$ is the orthogonal sum $\mathbb{R}^3 = Y \oplus Y^\perp$. □

Example 3.42 (orthogonal complement in $\Pi_1([0,1])$)
Let $\Pi_1([0,1]) = \mathrm{span}\,\{1, t\}$ be the real linear space of polynomials of degree ≤ 1 on $[0,1]$ with real coefficients, with the inner product

$$\langle f, g \rangle_{L_2([0,1])} := \int_0^1 f(t)\, g(t)\, \mathrm{d}t,$$

and let $Y := \Pi_0([0,1]) = \mathrm{span}\,\{1\}$ be the subspace of constant polynomials. Find $Y^\perp$.

Solution: Since $\dim P_1([0,1]) = 2$ and $\dim Y = 1$, we know that $\dim Y^\perp = 1$. We find a linear polynomial that is orthogonal to the constant polynomial 1 (and hence to all constant polynomials).

$$\langle a + b\,t, 1 \rangle_{L_2([0,1])} = \int_0^1 \big(a + b\,t\big)\, 1\, \mathrm{d}t = \left[a\,t + \frac{b}{2}\,t^2 \right]_0^1 = a + \frac{b}{2} = 0.$$

Thus we may choose $a = 1$ and $b = -2$. Then $p(t) := 1 - 2\,t$ is orthogonal to Y and

$$Y^\perp = \mathrm{span}\,\{p(t) = 1 - 2\,t\} = \{\alpha\, p(t) = \alpha - 2\,\alpha\, t \,:\, \alpha \in \mathbb{R}\}.$$

The inner product space $\Pi_1([0,1])$ is the orthogonal sum $\Pi_1([0,1]) = Y \oplus Y^\perp$. □

Exercise 43 *Consider $\mathbb{R}^3$ with the Euclidean inner product $\langle \mathbf{x}, \mathbf{y} \rangle_2 = \sum_{j=1}^{3} x_j\, y_j$. Let*

$$Y := \mathrm{span}\, \left\{ (2,0,1)^T, (1,1,-2)^T \right\}.$$

Determine the orthogonal complement $Y^\perp$.

Exercise 44 *Let $H = \Pi_3([-1,1]) = \mathrm{span}\,\{1, t, t^2\}$ be the real linear space of polynomials of degree ≤ 2 on $[-1,1]$ with real coefficients, endowed with the inner product*

$$\langle f, g \rangle_{L_2([-1,1])} = \int_{-1}^1 f(t)\, g(t)\, \mathrm{d}t.$$

Let $Y = \mathrm{span}\,\{1, t^2\}$. Find $Y^\perp$. Show your work.

Definition 3.43 (projection and orthogonal projection operator)

(i) Let X be a linear space, and let $P : X \to X$ be a linear function such that $P^2 = P \circ P = P$. Then P is called a ***projection (operator) from*** *X* ***onto*** *$\mathrm{range}(P)$.*

(ii) Let X be an inner product space with inner product $\langle \cdot, \cdot \rangle$, and let P be a projection from X onto $\mathrm{range}(P)$. Then P is the ***orthogonal projection (operator) onto*** *$\mathrm{range}(P)$ if*

$$\langle Px, y \rangle = \langle x, Py \rangle \qquad \text{for all } x, y \in X.$$

Example 3.44 (projections for $\mathbb{R}^2$)

Let $\mathbb{R}^2$ be endowed with the usual Euclidean inner product $\langle \mathbf{x}, \mathbf{y} \rangle_2 := x_1\, y_1 + x_2\, y_2$.

(a) The function $P : \mathbb{R}^2 \to \mathbb{R}^2$,

$$P\mathbf{x} := \mathbf{A}\,\mathbf{x} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\,x_1 + x_2 \end{pmatrix}, \tag{3.2.9}$$

is a projection but not an orthogonal projection. Indeed $P^2\mathbf{x} = \mathbf{A}\,\mathbf{A}\,\mathbf{x}$ and

$$\mathbf{A}\,\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} = \mathbf{A}.$$

Hence $P^2\mathbf{x} = \mathbf{A}\,\mathbf{A}\,\mathbf{x} = \mathbf{A}\,\mathbf{x} = P\mathbf{x}$, and P is clearly a projection. However, from (3.2.9)

$$\langle P\mathbf{x}, \mathbf{y} \rangle_2 = 0 + (2\,x_1 + x_2)\,y_2 \neq 0 + x_2\,(2\,y_1 + y_2) = \langle \mathbf{x}, P\mathbf{y} \rangle_2,$$

which shows that P is not an orthogonal projection.

(b) The function $P : \mathbb{R}^2 \to \mathbb{R}^2$,

$$P\mathbf{x} := \mathbf{A}\,\mathbf{x} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, \tag{3.2.10}$$

is an orthogonal projection. Indeed $P^2\mathbf{x} = \mathbf{A}\,\mathbf{A}\,\mathbf{x}$ and

$$\mathbf{A}\,\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{A}.$$

Hence $P^2\mathbf{x} = \mathbf{A}\,\mathbf{A}\,\mathbf{x} = \mathbf{A}\,\mathbf{x} = P\mathbf{x}$, and P is clearly a projection. Moreover, from (3.2.10)

$$\langle P\mathbf{x}, \mathbf{y} \rangle_2 = 0 + x_2\, y_2 = \langle \mathbf{x}, P\mathbf{y} \rangle_2 \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^2,$$

which shows that P is an orthogonal projection. □

Lemma 3.45 (elementary properties of projection operators)

(i) Let X be a linear space, and let $P : X \to X$ be a projection operator. Restricted to the linear space $\mathrm{range}(P)$, the projection P is the identity operator, that is, $Px = x$ for all $x \in \mathrm{range}(P)$.

(ii) Let H be an inner product space, and let $P : H \to Y$ be an orthogonal projection onto a subspace Y. Then $Px = \mathcal{O}$ for any x that is orthogonal to the subspace Y.

Exercise 45 *Give the proof of Lemma 3.45.*

Lemma 3.46 (best approx. in closed subspace Y is orthogonal proj. onto Y) *Suppose H is a Hilbert space and $Y \subset H$ a **closed subspace**. The function $P : H \to Y$ that assigns to every $x \in H$ its **best approximation** x^* **in** Y, that is $Px := x^*$, is the **orthogonal projection** onto Y.*

In the next section we will learn more theory that allows us to compute the orthogonal projection onto a given closed subspace in a convenient way.

Proof of Lemma 3.46: Let $x \in H$ be an arbitrary element and let $Px := x^*$ be its best approximation in Y.

The best approximation of x in Y is the unique element x^* in Y such that $\mathrm{dist}(x, Y) = \|x - x^*\|$. Thus $P(x^*) = P(P(x))$ is the unique element in Y such that $\mathrm{dist}(x^*, Y) = \|P(x^*) - x^*\|$. But since $x^* \in Y$, we have $\mathrm{dist}(x^*, Y) = 0$, and hence $P(x^*) = x^*$. Thus, from as $x^* = Px$, we find $P^2x = Px$, and as $x \in H$ was arbitrary we have $P^2 = P$. Therefore we know that the function P is a projection, and we also know that $\mathrm{range}(P) = Y$. It remains to show that the projection is an orthogonal projection.

From Theorem 3.32 we know that $x - x^* = x - Px$ is orthogonal to Y. Hence for all $x, z \in H$

$$\begin{aligned}
\langle Px, z\rangle &= \big\langle Px, Pz + (z - Pz)\big\rangle \\
&= \big\langle Px, Pz\big\rangle + \underbrace{\big\langle Px, (z - Pz)\big\rangle}_{=0} \\
&= \big\langle x - (x - Px), Pz\big\rangle \\
&= \big\langle x, Pz\big\rangle - \underbrace{\big\langle x - Px, Pz\big\rangle}_{=0} \\
&= \langle x, Pz\rangle,
\end{aligned}$$

where we have used that $Px \perp (z - Pz)$ and $(x - Px) \perp Pz$. Hence $\langle Px, z\rangle = \langle x, Pz\rangle$ for all $x, z \in H$, which verifies that the projection is an orthogonal projection. □

Exercise 46 *Let H be a Hilbert space and let $P : H \to Y$ be an orthogonal projection onto a closed subspace $Y \subset H$. Show that the functions $I - P : H \to H$ (where $I : H \to H$ is the identity map $Ix = x$), $(I - P)(x) = x - Px$, is also an orthogonal projection. What is the range of $I - P$?*

We end this section with a final look at the orthogonal complement of an arbitrary set M.

Lemma 3.47 *Let H be a Hilbert space and let M_1, M_2, and M be subsets of H.*

(i) *If $M_1 \subset M_2$ then $M_2^\perp \subset M_1^\perp$;*

(ii) $M^\perp = \overline{M}^\perp$;

(iii) $M^\perp = (\mathrm{span}\, M)^\perp$.

We note that from (ii) and (iii) in Lemma 3.47, for any subset M in H

$$M^\perp = (\overline{\operatorname{span} M})^\perp.$$

If the linear space $\operatorname{span} M$ is finite dimensional, then $\operatorname{span} M = \overline{\operatorname{span} M}$ and taking the closure of $\operatorname{span} M$ gives now new elements. It is for the case of **infinite dimensional spaces** $\operatorname{span} M$ where taking the closure usually makes a huge difference and where $\operatorname{span} M$ is usually a true subset of $\overline{\operatorname{span} M}$. If $\operatorname{span} M$ is infinite dimensional then taking the closure $\overline{\operatorname{span} M}$ means that we add all accumulation points, that is, we add all limits of sequences in $\operatorname{span} M$ that converge in H.

Proof of Lemma 3.47:

(i) As $M_2^\perp = \{x \in H \,:\, \langle x, y\rangle = 0 \text{ for all } y \in M_2\}$ and as $M_1 \subset M_2$, any element in $x \in M_2^\perp$ also satisfies

$$\langle x, y\rangle = 0 \qquad \text{for all } y \in M_1 \subset M_2.$$

Hence x is also in $M_1^\perp$ which shows that $M_2^\perp \subset M_1^\perp$.

(ii) As $M \subset \overline{M}$, (i) implies that $\overline{M}^\perp \subset M^\perp$. It remains to show that $M^\perp \subset \overline{M}^\perp$. Let $x_0 \in M^\perp$, that is, $\langle x_0, x\rangle = 0$ for all $x \in M$. As the set $\overline{M}$ is the union of M with the set of all accumulation points of M, it is enough to show that $\langle x_0, x\rangle = 0$ for any accumulation point x of M. For any accumulation point x of M, we can find a sequence $(x_n)_{n\in\mathbb{N}} \subset M$ that converges to x, that is, $\lim_{n\to\infty} \|x_n - x\| = 0$. Then $\langle x_n, x_0\rangle = 0$ for all $n \in \mathbb{N}$, and by the continuity of the inner product (see Lemma 3.10)

$$0 = \lim_{n\to\infty} \underbrace{\langle x_n, x_0\rangle}_{=0} = \langle x, x_0\rangle.$$

Thus $x \perp x_0$ for any accumulation point x of M, and hence $x_0 \in \overline{M}^\perp$. As $x_0 \in M^\perp$ was arbitrary, this shows that $M^\perp \subset \overline{M}^\perp$.

(iii) As $M \subset \operatorname{span}(M)$, we know from (i) that $(\operatorname{span} M)^\perp \subset M^\perp$. It remains to prove that also $M^\perp \subset (\operatorname{span} M)^\perp$. By definition, $x \in M^\perp$ if and only if

$$\langle x, y\rangle = 0, \qquad \text{for all } y \in M.$$

By the linearity of the scalar product in each variable, $x \in M^\perp$ is also orthogonal to any linear combination of vectors from M, that is, if $y_1, y_2, \ldots, y_n \in M$, then also

$$\left\langle x, \sum_{j=1}^{n} \alpha_j\, y_j \right\rangle = \sum_{j=1}^{n} \overline{\alpha_j}\, \langle x, y_j\rangle = 0$$

for any choice of $\alpha_1, \alpha_2, \ldots, \alpha_n$ and any $n \in \mathbb{N}$. Hence $x \perp \operatorname{span} M$, that is, $x \in (\operatorname{span} M)^\perp$. As $x \in M^\perp$ was arbitrary, we see that $M^\perp \subset (\operatorname{span} M)^\perp$.

Thus we have proved properties (i) to (iii). □

Lemma 3.48 (characterisation of a dense $\operatorname{span} M$ **in a Hilbert space)**
Let M *be a subset of a Hilbert space* H*. Then* $\operatorname{span} M$ *is dense in* H *(that is,* $\overline{\operatorname{span} M} = H$*) if and only if* $M^\perp = \{\mathcal{O}\}$*.*

Proof of Lemma 3.48: $\Rightarrow$: Let $Y = \operatorname{span} M$ be dense in H. Let $x \in M^\perp$ be arbitrary. Because $\operatorname{span} M$ is dense in H, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{span} M$ such that $x_n \to x$ as $n \to \infty$. On the other hand, from Lemma 3.47 (iii), we know $M^\perp = (\operatorname{span} M)^\perp$ so that $x \perp \operatorname{span}(M)$ and, in particular, $\langle x_n, x\rangle = 0$ for all $n \in \mathbb{N}$. Thus from the continuity of the inner product (see Lemma 3.10)

$$0 = \lim_{n\to\infty} \langle x_n, x\rangle = \langle x, x\rangle = \|x\|^2.$$

From $\|x\| = 0$ we know hat $x = \mathcal{O}$, Hence $M^\perp$ contains only the zero vector, that is, $M^\perp = \{\mathcal{O}\}$.

$\Leftarrow$: Assume that $M^\perp = \{\mathcal{O}\}$. Then, from Lemma 3.47 (iii), $(\operatorname{span} M)^\perp = M^\perp = \{\mathcal{O}\}$. Furthermore by Lemma 3.47 (ii) we find $(\overline{\operatorname{span} M})^\perp = (\operatorname{span} M)^\perp = \{\mathcal{O}\}$. As $\overline{\operatorname{span} M}$ is a closed linear subspace, by Theorem 3.40, we have the orthogonal sum

$$H = \overline{\operatorname{span} M} \oplus (\overline{\operatorname{span} M})^\perp = \overline{\operatorname{span} M} \oplus \{\mathcal{O}\},$$

which implies $\overline{\operatorname{span} M} = H$, that is, $\operatorname{span} M$ is dense in H. □

3.3 Orthonormal Sets and Orthogonal Projection

In this section we come back to the concept of orthogonality and discuss orthonormal (that is, orthogonal and normalised) sets of vectors and the orthogonal projection onto a closed subspace.

Definition 3.49 (orthogonal subset and orthonormal subset)
Let X *be an inner product space with inner product* $\langle \cdot, \cdot \rangle$*.*

(i) A subset $M \subset X$ *is called* ***orthogonal*** *if all its elements are pairwise orthogonal.*

(ii) A subset $M \subset X$ *is called* ***orthonormal*** *if it is orthogonal and all its elements have norm* 1*, that is, for all* $x, y \in M$ *we have*

$$\langle x, y\rangle = \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

Example 3.50 (canonical basis in $\mathbb{C}^d$**)**
Consider $\mathbb{C}^d$ with the Euclidean inner product $\langle \mathbf{x}, \mathbf{y}\rangle_2 = \sum_{k=1}^d x_k\, \overline{y_k}$. Then the **canonical basis**

$$\mathbf{e}_1 = (1,0,0,\ldots,0,0)^T, \quad \mathbf{e}_2 = (0,1,0,\ldots,0,0)^T, \quad \ldots, \quad \mathbf{e}_d = (0,0,0,\ldots,0,1)^T,$$

is an orthonormal set in $\mathbb{C}^d$. □

In the sequel, we will mainly be concerned with **countable** orthonormal sets.

Lemma 3.51 (members of an orthonormal set are linearly independent)
Let X be an inner product space. An orthonormal set $M \subset X$ is linearly independent.

Proof: We have to show that any finite subset $\{e_1, \ldots, e_n\}$ of the given orthonormal set M is linearly independent. Consider the equation

$$\sum_{k=1}^{n} \alpha_k \, e_k = \mathcal{O}.$$

Taking the inner product with e_ℓ, $\ell = 1, 2, \ldots, n$, shows

$$\left\langle \sum_{k=1}^{n} \alpha_k \, e_k, e_\ell \right\rangle = \sum_{k=1}^{n} \alpha_k \underbrace{\langle e_k, e_\ell \rangle}_{= \delta_{k,\ell}} = \alpha_\ell = \langle \mathcal{O}, e_\ell \rangle = 0 \qquad \text{for all } \ell = 1, 2, \ldots, n.$$

Thus we see that $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, and we have verified that $e_1, e_2, \ldots, e_n$ are linearly independent. □

It is also important to know that any countable linearly independent set can be 'converted' into an orthonormal set with the **Gram-Schmidt orthonormalisation** procedure.

Lemma 3.52 (Gram-Schmidt orthonormalisation)
Let X be an inner product space and let M be a subset $M := \{\phi_1, \phi_2, \ldots, \}$ of linearly independent elements of X. Then there exists an orthonormal set $\widetilde{M} = \{\psi_1, \psi_2, \ldots, \}$ such that $\operatorname{span} M = \operatorname{span}(\widetilde{M})$.

Proof: The proof is by inductive construction. First set $\psi_1 := \phi_1 / \|\phi_1\|$. Then, suppose that n orthonormal vectors $\psi_1, \psi_2, \ldots, \psi_n$ spanning the same space as $\phi_1, \phi_2, \ldots, \phi_n$, that is, $\operatorname{span}\{\psi_1, \psi_2, \ldots, \psi_n\} = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_n\}$, have been constructed. Then, we set

$$\widetilde{\psi}_{n+1} = \phi_{n+1} - \sum_{k=1}^{n} \langle \phi_{n+1}, \psi_k \rangle \, \psi_k.$$

Taking the inner product with ψ_j for $1 \leq j \leq n$ shows orthogonality:

$$\begin{aligned} \langle \widetilde{\psi}_{n+1}, \psi_j \rangle &= \left\langle \phi_{n+1} - \sum_{k=1}^{n} \langle \phi_{n+1}, \psi_k \rangle \, \psi_k \, , \psi_j \right\rangle \\ &= \langle \phi_{n+1}, \psi_j \rangle - \sum_{k=1}^{n} \langle \phi_{n+1}, \psi_k \rangle \underbrace{\langle \psi_k, \psi_j \rangle}_{= \delta_{j,k}} \\ &= \langle \phi_{n+1}, \psi_j \rangle - \langle \phi_{n+1}, \psi_j \rangle \; = \; 0. \end{aligned}$$

Hence, setting $\psi_{n+1} := \widetilde{\psi}_{n+1} / \|\widetilde{\psi}_{n+1}\|$ gives the new orthonormal element, such that we have $\operatorname{span}\{\psi_1, \psi_2, \ldots, \psi_n, \psi_{n+1}\} = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_n, \phi_{n+1}\}$. □

From now on, we will focus on subsets that $\{e_1, e_2, \ldots, \}$ that are **orthonormal**.

> **Theorem 3.53 (orthogonal projection onto finite dimensional subspace)**
> *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\{e_1, e_2, \ldots, e_n, \ldots\}$ be an* ***orthonormal set****. Define $U_n := \operatorname{span}\{e_1, \ldots, e_n\}$, and let $P_n : H \to U_n$ be the* ***orthogonal projection onto*** *U_n. Then for every $x \in H$,*
>
> *(i)* $P_n x = \sum_{k=1}^{n} \langle x, e_k \rangle \, e_k$,
>
> *(ii)* $\|P_n x\|^2 = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$,
>
> *(iii)* $\|x - P_n x\|^2 = \|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2$.

Proof of Theorem 3.53: As P_n is an orthogonal projection onto U_n, we know that

$$P_n x = \sum_{k=1}^{n} \alpha_k \, e_k. \tag{3.3.1}$$

with some coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$. Taking the inner product with e_ℓ, $\ell = 1, 2, \ldots, n$, on both sides, gives

$$\langle P_n x, e_\ell \rangle = \left\langle \sum_{k=1}^{n} \alpha_k \, e_k, e_\ell \right\rangle = \sum_{k=1}^{n} \alpha_k \underbrace{\langle e_k, e_\ell \rangle}_{= \delta_{k,\ell}} = \alpha_\ell.$$

As P_n is an orthogonal projection onto U_n, we have $P_n e_k = e_k$ for all $k = 1, 2, \ldots, n$ and $\langle P_n x, e_\ell \rangle = \langle x, P_n e_\ell \rangle = \langle x, e_\ell \rangle$ for $\ell = 1, 2, \ldots, n$, giving

$$\alpha_\ell = \langle P_n x, e_\ell \rangle = \langle x, e_\ell \rangle, \qquad \ell = 1, 2, \ldots, n. \tag{3.3.2}$$

Substituting (3.3.2) into (3.3.1) proves (i).

Using the orthonormality again, we see that (using (i))

$$\|P_n x\|^2 = \left\langle \sum_{k=1}^{n} \langle x, e_k \rangle \, e_k, \sum_{\ell=1}^{n} \langle x, e_\ell \rangle \, e_\ell \right\rangle = \sum_{k=1}^{n} \sum_{\ell=1}^{n} \langle x, e_k \rangle \, \overline{\langle x, e_\ell \rangle} \underbrace{\langle e_k, e_\ell \rangle}_{= \delta_{k,\ell}} = \sum_{k=1}^{n} |\langle x, e_k \rangle|^2,$$

which proves (ii).

Finally, from the properties of an orthogonal projection

$$\langle x - P_n x, P_n x \rangle = \langle x, P_n x \rangle - \langle P_n x, P_n x \rangle = \langle x, P_n x \rangle - \langle x, P_n^2 x \rangle = \langle x, P_n x \rangle - \langle x, P_n x \rangle = 0,$$

where we have used $P_n^2 = P_n$ in the second last step. Thus $x - P_n x$ is orthogonal to $P_n x$. Hence, Pythagoras theorem gives

$$\|x - P_n x\|^2 + \|P_n x\|^2 = \|x\|^2 \qquad \Leftrightarrow \qquad \|x - P_n x\|^2 = \|x\|^2 - \|P_n x\|^2,$$

and this together with (ii) leads to (iii). □

Definition 3.54 (Fourier coefficients)
Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $\{e_1, e_2, \ldots\}$ be an orthonormal set. Let $x \in H$. The numbers

$$\langle x, e_k\rangle, \qquad k = 1, 2, \ldots,$$

are called the ***Fourier coefficients of*** *x with respect to the orthonormal set $\{e_1, e_2, \ldots\}$.*

From Theorem 3.53 (ii) and (iii) we have $\|P_n x\|^2 = \|x\|^2 - \|x - P_n x\|^2$, which implies that $\|P_n x\|^2 \leq \|x\|^2$. As a bounded increasing sequence in $\mathbb{R}$ converges and

$$\|P_n x\|^2 = \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 \leq \|x\|^2,$$

we see that $\lim_{n\to\infty} \|P_n x\|^2$ exists and satisfies

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 = \lim_{n\to\infty} \|P_n x\|^2 \leq \|x\|^2.$$

This relation is called the **Bessel inequality**. Hence, we have proved the first part of the following theorem.

Theorem 3.55 (Bessel inequality and series expansion of projection onto $\overline{\text{span}\, M}$)
Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $M = \{e_1, e_2, \ldots\}$ be an orthonormal set. Then the following holds true:

(i) ***Bessel inequality:*** *For any $x \in X$*

$$\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \leq \|x\|^2. \tag{3.3.3}$$

(ii) The series

$$y := \sum_{k=1}^{\infty} \langle x, e_k\rangle\, e_k \tag{3.3.4}$$

is convergent, and the vector y coincides with the ***orthogonal projection*** *Px* ***of*** *x* ***on the closed subspace*** *$\overline{\text{span}\,(M)}$. In other words, Px is the* ***best approximation of*** *x* ***in*** *$\overline{\text{span}\,(M)}$.*

Proof of Theorem 3.55: The Bessel inequality was derived before we stated the theorem. It remains to prove (ii). As a preparation we verify that $\overline{\text{span}\, M}$ is a closed subspace of H. By definition of the closure, $\overline{\text{span}\, M}$ is closed, and $\text{span}\, M$ is by definition a subspace. It remains to show that $\overline{\text{span}\, M}$ is closed under addition and scalar multiplication. For any $x, y \in \overline{\text{span}\, M}$, there exist sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subset \text{span}\, M$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. As, for any $\alpha, \beta \in \mathbb{K}$, the sequence $(\alpha\, x_n + \beta\, y_n)_{n\in\mathbb{N}}$ is in $\text{span}\, M$ and as

$$\big\|(\alpha\, x+\beta\, y)-(\alpha\, x_n+\beta\, y_n)\big\| = \big\|\alpha\,(x-x_n)+\beta\,(y-y_n)\big\| \leq |\alpha|\, \|x-x_n\|+\beta\, \|y-y_n\| \to 0 \quad \text{for } n \to \infty,$$

we see that $\alpha\, x + \beta\, y$ is also in $\overline{\operatorname{span} M}$. Hence $\overline{\operatorname{span} M}$ is a subspace.

To verify (ii), for a given $x \in H$, we define the sequence $(y_n)_{n\in\mathbb{N}}$ by

$$y_n := P_n x = \sum_{k=1}^{n} \langle x, e_k\rangle\, e_k,$$

where P_n is as before the orthogonal projection onto $U_n = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. This sequence is a Cauchy sequence, since for $n < m$

$$\|y_m - y_n\|^2 = \left\| \sum_{k=n+1}^{m} \langle x, e_k\rangle\, e_k \right\|^2 = \sum_{k=n+1}^{m} \sum_{\ell=n+1}^{m} \langle x, e_k\rangle\, \overline{\langle x, e_\ell\rangle}\, \underbrace{\langle e_k, e_\ell\rangle}_{=\,\delta_{k,\ell}} = \sum_{k=n+1}^{m} |\langle x, e_k\rangle|^2$$

From the Bessel inequality and the previous formula we can now conclude that for every $\epsilon > 0$ there exists an $N = N(\epsilon) \in \mathbb{N}$ such that $\|y_m - y_n\| < \epsilon$ for all $n, m \geq N$. Thus $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

As $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space H it converges to some limit which we denote by $y \in H$ and which is given by (3.3.4). As $y_n \in \operatorname{span} M$ for all $n \in \mathbb{N}$, we know that $y = \lim_{n\to\infty} y_n \in \overline{\operatorname{span} M}$.

Finally, due to the fact that $\langle e_k, e_j\rangle = \delta_{j,k}$, we can conclude that

$$\langle y - x, e_j\rangle = \left\langle \sum_{k=1}^{\infty} \langle x, e_k\rangle\, e_k - x, e_j \right\rangle = \sum_{k=1}^{\infty} \langle x, e_k\rangle\, \underbrace{\langle e_k, e_j\rangle}_{=\,\delta_{j,k}} - \langle x, e_j\rangle = \langle x, e_j\rangle - \langle x, e_j\rangle = 0.$$

This proves that $(y - x) \perp M$ and hence $(x - y) \perp \overline{\operatorname{span}(M)}$ (see Lemma 3.47). As $\overline{\operatorname{span} M}$ is a closed subspace, Theorem 3.32 shows that y is the best approximation of x in $\overline{\operatorname{span} M}$. □

We discuss an example to get a better feeling for these new concepts.

Example 3.56 (complex trigonometric basis polynomials)
In Example 3.19 we already encountered the complex trigonometric basis polynomials and saw that they form an orthogonal set. By choosing the normalisation factors in (3.1.29) as $\alpha_k = (\sqrt{2\pi})^{-1}$ the complex trigonometric basis polynomials become orthonormal. Thus the set

$$M = \{e_k\}_{k\in\mathbb{Z}} = \{\ldots, e_{-2}, e_{-1}, e_0, e_1, e_2, \ldots\},$$

where

$$e_k(x) := \frac{1}{\sqrt{2\pi}}\, e^{ikx}, \qquad x \in [-\pi, \pi], \qquad k \in \mathbb{Z},$$

forms an orthonormal set in the space $L_2([-\pi, \pi])$ with the inner product

$$\langle f, g\rangle_{L_2[-\pi,\pi]} := \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, \mathrm{d}x.$$

Introducing the space $U_n := \operatorname{span}\{e_{-n},\ldots,e_{-1},e_0,e_1,\ldots,e_n\}$ of trigonometric polynomials of degree $\le n$, the orthogonal projection onto U_n is given by

$$\begin{aligned}(P_nf)(x) \;:=\; \sum_{k=-n}^{n}\langle f,e_k\rangle_{L_2([-\pi,\pi])}\,e_k(x) &= \sum_{k=-n}^{n}\left(\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}f(y)\,e^{-iky}\,\mathrm{d}y\right)\frac{1}{\sqrt{2\pi}}\,e^{ikx}\\ &= \frac{1}{2\pi}\sum_{k=-n}^{n}\left(\int_{-\pi}^{\pi}f(y)\,e^{-iky}\,\mathrm{d}y\right)e^{ikx}.\end{aligned}$$

For a given $f\in L_2([-\pi,\pi])$, the function P_nf is the best approximation of the function f in the space of U_n of trigonometric polynomials of degree $\le n$.

Let us compute P_nf for the example $f(x)=x$. For $k=0$, we have $e_0(x)=(\sqrt{2\pi})^{-1}\,e^{i0x}=(\sqrt{2\pi})^{-1}$. Hence the Fourier coefficient $\langle f,e_0\rangle_{L_2([-\pi,\pi])}$ is given by

$$\langle f,e_0\rangle_{L_2([-\pi,\pi])}=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}x\,1\,\mathrm{d}x=\frac{1}{\sqrt{2\pi}}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi}=\frac{1}{\sqrt{2\pi}}\left[\frac{\pi^2}{2}-\frac{(-\pi)^2}{2}\right]=0.$$

Using integration by parts and later-on Euler's formula, the Fourier coefficients with $k\neq 0$ are given by

$$\begin{aligned}\langle f,e_k\rangle_{L_2([-\pi,\pi])} \;=\; \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}x\,e^{-ikx}\,\mathrm{d}x &= \frac{1}{\sqrt{2\pi}}\left(\left[x\,\frac{e^{-ikx}}{-ik}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\frac{e^{-ikx}}{-ik}\,\mathrm{d}x\right)\\ &= \frac{1}{\sqrt{2\pi}}\left(\left[x\,\frac{e^{-ikx}}{-ik}\right]_{-\pi}^{\pi}-\left[\frac{e^{-ikx}}{(-ik)^2}\right]_{-\pi}^{\pi}\right)\\ &= \frac{1}{\sqrt{2\pi}}\left(\left[\pi\,\frac{e^{-ik\pi}}{-ik}-(-\pi)\,\frac{e^{ik\pi}}{-ik}\right]-\left[\frac{e^{-ik\pi}}{-k^2}-\frac{e^{ik\pi}}{-k^2}\right]\right)\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{-\pi}{ik}\left(e^{ik\pi}+e^{-ik\pi}\right)-\frac{1}{k^2}\left(e^{ik\pi}-e^{-ik\pi}\right)\right)\\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{-2\pi}{ik}\cos(k\pi)-\frac{2i}{k^2}\sin(k\pi)\right)\\ &= \frac{\sqrt{2\pi}\,i}{k}\,(-1)^k,\end{aligned}$$

where we used that $\sin(k\pi)=0$ and $\cos(k\pi)=(-1)^k$ for all $k\in\mathbb{Z}$. Thus the orthogonal projection P_nf of $f(x)=x$ onto the space of trigonometric polynomials of degree n is given by

$$\begin{aligned}(P_nf)(x) &= \sum_{k=-n}^{-1}\frac{\sqrt{2\pi}\,i}{k}\,(-1)^k\,\frac{1}{\sqrt{2\pi}}\,e^{ikx}+\sum_{k=1}^{n}\frac{\sqrt{2\pi}\,i}{k}\,(-1)^k\,\frac{1}{\sqrt{2\pi}}\,e^{ikx}\\ &= \sum_{k=-n}^{-1}\frac{i}{k}\,(-1)^k\,e^{ikx}+\sum_{k=1}^{n}\frac{i}{k}\,(-1)^k\,e^{ikx}\\ &= \sum_{k=1}^{n}\frac{i}{-k}\,(-1)^{-k}\,e^{-ikx}+\sum_{k=1}^{n}\frac{i}{k}\,(-1)^k\,e^{ikx}\end{aligned}$$

$$= \sum_{k=1}^{n} \frac{i}{k}\,(-1)^k \left(e^{ikx} - e^{-ikx}\right)$$

$$= \sum_{k=1}^{n} \frac{i}{k}\,(-1)^k\, 2i\, \sin(kx)$$

$$= \sum_{k=1}^{n} \frac{2\,(-1)^{k+1}}{k}\, \sin(kx),$$

where we have again used Euler's formula.

Finally we verify the Bessel inequality for this example:

$$\sum_{k=-n}^{n} \left|\langle f, e_k\rangle_{L_2([-\pi,\pi])}\right|^2 = \sum_{k=-n}^{-1} \left|\frac{\sqrt{2\pi}\,i}{k}\,(-1)^k\right|^2 + \sum_{k=1}^{n} \left|\frac{\sqrt{2\pi}\,i}{k}\,(-1)^k\right|^2 = \sum_{k=-n}^{-1} \frac{2\pi}{k^2} + \sum_{k=1}^{n} \frac{2\pi}{k^2} = 4\pi \sum_{k=1}^{n} \frac{1}{k^2}$$

and

$$\|f\|^2_{L_2([-\pi,\pi])} = \int_{-\pi}^{\pi} |f(x)|^2\, \mathrm{d}x = \int_{-\pi}^{\pi} x^2\, dx = \left[\frac{1}{3}\, x^3\right]_{-\pi}^{\pi} = \frac{2}{3}\,\pi^3.$$

As $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$ (which is not obvious and was determined with Maple), we see that Bessel's inequality is indeed true for this example

$$\|P_n f\|^2_{L_2([-\pi,\pi])} = \sum_{k=-n}^{n} \left|\langle f, e_k\rangle_{L_2([-\pi,\pi])}\right|^2 = 4\pi \sum_{k=1}^{n} \frac{1}{k^2} \leq 4\pi \sum_{k=1}^{\infty} \frac{1}{k^2} = 4\pi\,\frac{\pi^2}{6} = \frac{2\,\pi^3}{3} = \|f\|^2_{L_2([-\pi,\pi])}.$$

We note that for this example we have $= \sum_{k=-\infty}^{\infty} \left|\langle f, e_k\rangle_{L_2([-\pi,\pi])}\right|^2 = \|f\|^2_{L_2([-\pi,\pi])}$, and this leads to the question whether we also have (in the $L_2([-\pi,\pi])$ sense)

$$\sum_{k=-\infty}^{\infty} \langle f, e_k\rangle_{L_2([-\pi,\pi])}\, e_k = f.$$

This relation is indeed true as we will see in Chapter 4. □

Remark 3.57 (representation of $x \in H$ as series w.r.t. orthonormal set)
The last example raises the question that will be addressed in the next section: Suppose we have a Hilbert space H and a countable orthonormal set $M = \{e_1, e_2, \ldots\}$ in H. Under what conditions on M do we have that

$$x = Px := \sum_{k=1}^{\infty} \langle x, e_k\rangle\, e_k \tag{3.3.5}$$

and

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \tag{3.3.6}$$

for all $x \in H$? – From Theorem 3.55 we know that Px in (3.3.5) is the orthogonal projection onto the closed subspace $\overline{\operatorname{span} M}$, or, in other words, Px is the best approximation of x in $\overline{\operatorname{span} M}$. Hence we have the equality $x = Px$ in (3.3.5) if and only $H = \overline{\operatorname{span} M}$. If this is the case then (3.3.6) follows automatically.

3.4 Schauder Basis and Orthonormal Basis

Finally we introduce the concept of a **Schauder basis** of an infinite dimensional space. A particular case is the orthonormal Schauder basis in a Hilbert space, usually called **orthonormal basis** or also **complete orthonormal system**. An orthonormal basis for a Hilbert space H is exactly the case or an orthonormal set $M = \{e_1, e_2, \ldots\}$ in H, where $\overline{\operatorname{span} M} = H$ and where consequently every $x \in H$ has a series representation (3.3.5); see also Remark 3.57.

If a Hilbert space H has an orthonormal basis $M = \{e_1, e_2, \ldots\}$, then every $x \in X$ can be expanded into a **Fourier series**

$$x = \sum_{k=0}^{\infty} c_k \, e_k, \tag{3.4.1}$$

with uniquely determined Fourier coefficients $c_k = \langle x, e_k \rangle$, $k \in \mathbb{N}$, and the Fourier series (3.4.1) converges with respect to the norm $\| \cdot \|$ of H. The function that maps $x \in H$ onto the sequence $(c_k)_{k \in \mathbb{N}}$ of its Fourier coefficients defines a **bijection between** H **and** $\ell_2(\mathbb{N})$. Thus we can either study $x \in H$ by studying it directly or by studying its sequence $(c_k)_{k \in \mathbb{N}}$ of Fourier coefficients in $\ell_2(\mathbb{N})$.

At the end of the section we come back to the concept of the orthogonal sum and discuss the concept of the **orthogonal sum of several (or even infinitely many) subspaces**. We also consider the projections onto these subspaces. These ideas will play an important rule in Chapter 5, when we discuss multiresolution analysis.

We begin by introducing a more general concept of a basis than the one you have encountered in linear algebra.

Definition 3.58 (Schauder basis for a Banach space)
Let X be a Banach space with norm $\| \cdot \|$, and let $M = \{\phi_1, \phi_2, \ldots\}$ be a countable subset of H. Then M is called a ***Schauder basis*** *of H, if*

(i) M is linearly independent, and

(ii) for every $x \in H$, there exists a sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n \in \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ for each $n \in \mathbb{N}$, such that

$$\|x - x_n\| \to 0 \qquad \text{as } n \to \infty.$$

Equivalently, M is a Schauder basis of X if every $x \in X$ has a ***unique*** *representation*

$$x = \sum_{k=1}^{\infty} c_k \, \phi_k,$$

where the convergence is, of course, in the $\| \cdot \|$ sense.

Exercise 47 *Show that the two characterisations of a Schauder basis given in Definition 3.58 are equivalent.*

Now, let H be a Hilbert space, and let $\{\phi_1, \phi_2, \ldots\}$ be a countable linearly independent set in H. Then we know by the **Gram-Schmidt orthonormalisation procedure** (see Lemma 3.52) that there exists an orthonormal set $\{\psi, \psi_2, \ldots\}$ such that

$$\operatorname{span}\{\psi_1, \psi_2, \ldots, \psi_n\} = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_n\} \qquad \text{for all } n \in \mathbb{N}.$$

Thus we can restrict ourselves to **orthonormal countable sets** $M = \{e_1, e_2, \ldots\}$. Furthermore, let $P_n : H \to U_n$ denote the orthogonal projection onto $U_n = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. Consider $x \in H$. As $P_n x$ is also the best approximation of x in U_n we have

$$\|x - P_n x\| \leq \|x - x_n\| \to 0 \qquad \text{as } n \to \infty,$$

where x_n is from the sequence $(x_n)_{n\in\mathbb{N}}$ in Definition 3.58. Hence, we can characterise an orthonormal Schauder basis as follows.

Lemma 3.59 (characterisation of orthonormal Schauder basis)
Let H be a Hilbert space, and let $M = \{e_1, e_2, \ldots\}$ be a countable ***orthonormal*** *subset of H. Denote by $P_n : H \to U_n$ the orthogonal projection onto $U_n := \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. The set M forms a Schauder basis if and only if $\lim_{n\to\infty} \|P_n x - x\| = 0$ for all $x \in H$.*

As the orthogonal projection $P_n x$ of x onto U_n is given by

$$P_n(x) = \sum_{k=1}^{n} \langle x, e_k \rangle \, e_k,$$

the limit $\lim_{n\to\infty} \|P_n x - x\| = 0$ means that we have

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle \, e_k.$$

Definition 3.60 (orthonormal basis/complete orthonormal set)
Let H be a Hilbert space. An orthonormal set $M = \{e_1, e_2, \ldots\}$ is said to be an ***orthonormal basis*** *(or a* ***complete orthonormal set****) if $\operatorname{span} M$ is dense in H, that is, $\overline{\operatorname{span} M} = H$.*

We note that if $M = \{e_1, e_2, \ldots\}$ is an orthonormal basis, then from Lemma 3.59 and Theorem 3.55, we see that M is an orthonormal Schauder basis for H. From Theorem 3.55 (ii), we then obtain the following lemma.

Lemma 3.61 (representation by Fourier series)
Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $M = \{e_1, e_2, \ldots\}$ be an ***orthonormal basis****. Then every $x \in H$ can be represented by*

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle \, e_k. \tag{3.4.2}$$

The series in (3.4.2) is called the ***Fourier series of*** *x with respect to the orthonormal basis $M = \{e_1, e_2, \ldots\}$.*

Proof of Lemma 3.61: By the definition of an orthonormal basis, we have that $\overline{\operatorname{span} M} = H$. And by Theorem 3.55 we know that the series

$$\sum_{k=1}^{\infty} \langle x, e_k \rangle \, e_k$$

is the orthogonal projection $P : H \to \overline{\operatorname{span} M}$ onto $\overline{\operatorname{span} M} = H$. But the orthogonal projection onto H is the identity as $Px = Ix$ for all $x \in \overline{\operatorname{span} M} = H$. Hence (3.4.2) holds. □

Now, we have the following equivalent characterisations of an orthonormal basis in H.

Theorem 3.62 (Fourier series theorem)
Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Let $M = \{e_1, e_2, \ldots, \}$ be a ***countable orthonormal*** *set in H. Then, the following statements are equivalent:*

(i) M is a ***Schauder basis*** *of H.*

(ii) Every $x \in H$ can be represented by its ***Fourier series****:*

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle \, e_k.$$

(iii) For every $x \in H$, ***Parseval's identity*** *holds, that is,*

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 = \|x\|^2. \tag{3.4.3}$$

Proof: (i) ⇔ (ii): That (i) and (ii) are equivalent follows from Lemma 3.59 and Lemma 3.61.

The equivalence of (ii) and (iii) can be seen in the following way: From Theorem 3.53 (ii) and (iii), we know that, with $P_n x := \sum_{j=1}^{n} \langle x, e_j \rangle \, e_j$,

$$\|x\|^2 - \|x - P_n x\|^2 = \|P_n x\|^2 = \sum_{j=1}^{n} |\langle x, e_j \rangle|^2 \qquad \text{for all } x \in X. \tag{3.4.4}$$

(ii) ⇒ (iii): Assume now that (ii) (and equivalently (i)) holds: Then $\lim_{n\to\infty} \|x - P_n x\| = 0$. Thus taking the limit for $n \to \infty$ in (3.4.4) implies (iii).

(iii) ⇒ (ii): Assume (iii) is true. Then (3.4.3) implies

$$\lim_{n\to\infty} \left(\|x\|^2 - \sum_{k=1}^{n} |\langle x, e_k \rangle|^2 \right) = \lim_{n\to\infty} \left(\|x\|^2 - \|P_n x\|^2 \right) = 0,$$

and hence from (3.4.4)

$$\lim_{n\to\infty} \|x - P_n x\|^2 = \lim_{n\to\infty} \left(\|x\|^2 - \|P_n x\|^2 \right) = 0$$

which verifies that (ii) is true. □

Now, having a countable **orthonormal basis** in a Hilbert space H, we can identify the Hilbert space H with $\ell_2(\mathbb{N})$ in the following way: We know from (3.4.3) that, for an element $x \in H$, the sequence of Fourier coefficients $(\langle x, e_k\rangle)_{k\in\mathbb{N}}$ belongs to $\ell_2(\mathbb{N})$. On the other hand, for every sequence $(c_k)_{k\in\mathbb{N}}$ in ℓ_2, we can construct an element $x \in H$ having this sequence as its Fourier series, namely

$$x = \sum_{k=1}^{\infty} c_k\, e_k. \tag{3.4.5}$$

We note that, **if** this series converges with respect to $\|\cdot\|$, then we have from the orthonormality of the e_k, $k \in \mathbb{N}$, that $\langle x, e_\ell\rangle = \sum_{k=1}^{\infty} c_k \langle e_k, e_\ell\rangle = c_\ell$. The fact that for a given $(c_k)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N})$ the element (3.4.5) is well-defined is exactly the statement of the Riesz-Fischer theorem below.

Theorem 3.63 (Riesz-Fischer theorem)
Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $\{e_1, e_2, \ldots\}$ form an orthonormal basis in H. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence from $\ell_2(\mathbb{N})$. Then there exists an element $x \in H$ such that

$$\langle x, e_k\rangle = c_k \qquad \text{for all } k \in \mathbb{N},$$

and we have

$$x = \sum_{k=1}^{\infty} c_k\, e_k.$$

Proof of Theorem 3.63: Given $(c_k)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N})$ define

$$x := \sum_{k=1}^{\infty} c_k\, e_k = \lim_{n\to\infty} x_n, \qquad \text{where} \qquad x_n = \sum_{k=1}^{n} c_k\, e_k. \tag{3.4.6}$$

We know that x is in H, if the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. (Indeed, since H is complete, every Cauchy sequence converges with limit in H, and the limit of $(x_n)_{n\in\mathbb{N}}$, if it converges, is clearly x defined by (3.4.6).) For $\epsilon > 0$ there exists and $N = N(\epsilon) \in \mathbb{N}$ such that

$$\|x_n - x_m\|^2 = \left\| \sum_{k=m+1}^{n} c_k\, e_k \right\|^2 = \sum_{k=m+1}^{n} \sum_{\ell=m+1}^{n} c_k\, \overline{c_\ell}\, \langle e_k, e_\ell\rangle = \sum_{k=m+1}^{n} |c_k|^2 < \epsilon \quad \text{for all } n > m \geq N,$$

where we have used the fact that the e_k, $k = 1, 2, \ldots$, are mutually orthogonal and that $(\sum_{k=1}^{n} |c_k|^2)_{n\in\mathbb{N}}$ is a Cauchy sequence because $(c_k)_{k\in\mathbb{N}}$ is in $\ell_2(\mathbb{N})$. Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and its limit x, defined by (3.4.6), is in H.

Taking the inner product between x and e_m, $m \in \mathbb{N}$, we find

$$\langle x, e_m\rangle = \left\langle \sum_{k=1}^{\infty} c_k\, e_k, e_m \right\rangle = \sum_{k=1}^{\infty} c_k \langle e_k, e_m\rangle = c_m, \qquad m \in \mathbb{N},$$

where we were allowed to interchange the order of the sum and the inner product, because the inner product is continuous and because the series converges in H. □

From Theorem 3.62 and the Riesz-Fischer Theorem 3.63 above, we now have the following theorem. Remember that a bijection is a function that is both injective and and surjective (or in other words, one-to-one and onto).

Theorem 3.64 (bijection between $\ell_2(\mathbb{N})$ and Hilbert space with orthon. basis)
Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Assume that H has a ***countable orthonormal basis****. Then the function*

$$B : H \to \ell_2(\mathbb{N}), \qquad H(x) := \big(\langle x, e_k \rangle\big)_{n\in\mathbb{N}},$$

is a ***bijection*** *onto $\ell_2(\mathbb{N})$ and its inverse is given by*

$$B^{-1} : \ell_2(\mathbb{N}) \to H, \qquad B^{-1}\big((c_k)_{k\in\mathbb{N}}\big) := \sum_{k=1}^{\infty} c_k\, e_k.$$

Now we come back to Example 3.56 of the complex trigonometric polynomials.

Example 3.65 (complex trigonometric basis polynomials in $L_2([-\pi,\pi])$)
In the next chapter, we will see that the complex trigonometric basis polynomials

$$e_k(x) := \frac{1}{\sqrt{2\pi}}\, e^{ikx}, \qquad k \in \mathbb{Z},$$

form an orthonormal basis for $L_2([-\pi,\pi])$ endowed with the usual inner product

$$\langle f, g \rangle_{L_2([-\pi,\pi])} := \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, \mathrm{d}x.$$

Hence we know that for every function $f \in L_2([-\pi,\pi])$ we have (in the sense that the series converges with respect to the $\|\cdot\|_{L_2([-\pi,\pi])}$ norm)

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} c_k\, e^{ikx} \qquad \text{with} \qquad c_k := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, e^{ikx}\, \mathrm{d}x,$$

and Parseval's identity holds true

$$\int_{-\pi}^{\pi} |f(x)|^2\, \mathrm{d}x = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left| \int_{-\pi}^{\pi} f(x)\, e^{ikx}\, \mathrm{d}x \right|^2. \qquad \square$$

Exercise 48 *Consider the square wave function*

$$f(x) = \begin{cases} -1 & \text{if} \quad x \in (-\pi, 0), \\ 0 & \text{if} \quad x \in \{-\pi, 0, \pi\}, \\ 1 & \text{if} \quad x \in (0, \pi). \end{cases}$$

which is in $L_2([-\pi,\pi])$.

(a) *Compute the Fourier series of the square wave function f with respect to the orthonormal basis of complex trigonometric basis polynomials*

$$\left\{ e_k(x) = \frac{1}{\sqrt{2\pi}}\, e^{ikx} \;:\; k \in \mathbb{Z} \right\}.$$

Simplify the Fourier series as far as possible.

(b) *Verify Parseval's identity for the square wave function. (You may use Maple or another computing program to compute the limit of the series, if required.)*

Exercise 49 *Find an orthonormal basis for the Hilbert space $\ell_2(\mathbb{N})$ with the inner product*

$$\langle x, y \rangle_2 = \sum_{k=1}^{\infty} x_k \, \overline{y_k}, \qquad x = (x_k)_{k\in\mathbb{N}},\; y = (y_k)_{k\in\mathbb{N}} \in \ell_2(\mathbb{N}).$$

Verify that your orthonormal basis M has the properties of an orthonormal basis!

Finally, we come back to the definition of the direct sum (see Definition 3.37) and introduce the **direct sum of multiple subspaces**.

Definition 3.66 (direct sum of several subspaces)
Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$.

(i) *Let $Y_1, Y_2, \ldots, Y_n$ be subspaces of H. We say that H is the* ***direct sum*** *of the subspaces $Y_1, Y_2, \ldots, Y_n$, formally written as*

$$H = Y_1 \oplus Y_2 \oplus \cdots \oplus Y_n, \tag{3.4.7}$$

if every vector $x \in H$ has a ***unique*** *representation*

$$x = y_1 + y_2 + \cdots + y_n, \qquad y_j \in Y_j,\; j = 1, 2, \ldots, n. \tag{3.4.8}$$

(ii) *Let $Y_1, Y_2, \ldots$ be countable infinitely many subspaces of H. We say that H is the* ***direct sum*** *of the subspaces $Y_1, Y_2, \ldots$, formally written as*

$$H = Y_1 \oplus Y_2 \oplus \cdots = \bigoplus_{k=1}^{\infty} Y_k, \tag{3.4.9}$$

if every vector $x \in H$ has a ***unique*** *representation*

$$x = y_1 + y_2 + \cdots = \sum_{k=1}^{\infty} y_k, \qquad y_k \in Y_k,\; k \in \mathbb{N}. \tag{3.4.10}$$

If all subspaces Y_j are pair-wise orthogonal, we call (3.4.7) and (3.4.9) an ***orthogonal sum****. In this case the y_k, $k = 1, 2, \ldots, n$, in (3.4.8) and the y_k, $k \in \mathbb{N}$, in (3.4.10) are mutually orthogonal.*

In the case of an orthogonal sum, we have the following more specific statement.

Lemma 3.67 (projection on the closed subspaces of an orthogonal sum) *Let H be a Hilbert space that is the orthogonal sum of infinitely many closed subspaces $Y_1, Y_2, \ldots$, that is,*

$$H = \bigoplus_{k=1}^{\infty} Y_k.$$

For each $k = 1, 2, \ldots$, let $P_k : H \to Y_k$ be the orthogonal projection onto Y_k. Then

$$x = P_1 x + P_2 x + \ldots = \sum_{k=1}^{\infty} P_k x, \tag{3.4.11}$$

and for every x we have $P_k x \to \mathcal{O}$ as $k \to \infty$.

We note that an analogous statement to Lemma 3.67 holds if H is the orthogonal sum of only finitely many orthogonal closed subspaces $Y_1, Y_2, \ldots, Y_n$, that is,

$$H = \bigoplus_{k=1}^{n} Y_n.$$

If $P_k : H \to Y_k$ denotes the orthogonal projection onto Y_k, them for every $x \in H$

$$x = \sum_{k=1}^{n} P_k x.$$

This follows immediately from Lemma 3.67 by letting $Y_{n+1} = Y_{n+2} = \ldots = \{\mathcal{O}\}$.

Proof of Lemma 3.67: First we note that from the definition of the orthogonal sum the series (3.4.10) converges and hence

$$\|x\|^2 = \left\| \sum_{k=1}^{\infty} y_k \right\|^2 = \left\langle \sum_{j=1}^{\infty} y_j, \sum_{k=1}^{\infty} y_k \right\rangle = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \underbrace{\langle y_j, y_k \rangle}_{= \delta_{j,k}} = \sum_{k=1}^{\infty} \|y_k\|^2, \tag{3.4.12}$$

where we have used $\langle y_j, y_k \rangle = 0$ if $j \neq k$ as the the subspaces Y_j and Y_k are mutually orthogonal. From (3.4.12), we see that

$$\lim_{k\to\infty} \|y_k\|^2 = 0 \qquad \Leftrightarrow \qquad \lim_{k\to\infty} \|y_k\| = 0.$$

The statement follows if we can show that (3.4.11) holds true. Indeed then we have two sum representations (3.4.10) and (3.4.11) of x, and $P_k x \in Y_k$. As the representation (3.4.10) is unique, we can conclude that $y_k = P_k x$ and hence $\lim_{k\to\infty} \|P_k x\| = \lim_{k\to\infty} \|y_k\| = 0$.

It remains to verify (3.4.11). To do this we apply the orthogonal projection P_j to the representation (3.4.10) of x. Then

$$P_j x = P_j \left(\sum_{k=1}^{\infty} y_k \right) = \sum_{k=1}^{\infty} P_j y_k = P_j y_j = y_j,$$

where we have used $P_j y_k = 0$ if $k \neq j$ since Y_k is orthogonal to Y_j and that $P_j y_j = y_j$ since $y_j \in Y_j$. (We note that we are allowed to interchange the order of P_j and the sum as the series converges in H.) $\square$

The decomposition of a space into an infinite sum of orthogonal subspaces is quite important and we will use this later-on frequently when we discuss wavelets. We give one example here in the context of trigonometric functions.

Example 3.68 (complex trigonometric basis polynomials)
Let $\{e_k \,:\, k \in \mathbb{Z}\}$ denote the orthonormal set of complex trigonometric basis polynomials

$$e_k(x) := \frac{1}{\sqrt{2\pi}}\, e^{ikx}.$$

Even though we will only verify this in the next chapter we make now use of the fact that $M = \{e_k \,:\, k \in \mathbb{Z}\}$ is an orthonormal basis for $L_2([-\pi,\pi])$. We define the orthogonal subspaces $Y_1, Y_2, \ldots$ by

$$Y_0 := \text{span}\,\{e_0\} = \text{span}\left\{e_0(x) = \frac{1}{\sqrt{2\pi}}\right\}$$
$$Y_n := \text{span}\,\left\{e_k \,:\, 2^{n-1} \leq |k| < 2^n\right\} = \text{span}\left\{e_k(x) = \frac{1}{\sqrt{2\pi}}\, e^{ikx} \,:\, 2^{n-1} \leq |k| < 2^n\right\}, \qquad n \in \mathbb{N}.$$

Then we have the orthogonal sum

$$L_2([-\pi,\pi]) = \bigoplus_{n=0}^{\infty} Y_n,$$

and we may think of the orthogonal projection $P_n : L_2([-\pi,\pi]) \to Y_n$ as a **bandpass filter** as functions in Y_n contains only linear combinations of complex trigonometric basis polynomials with degrees from the band $2^{n-1} \leq |k| < 2^n$, that is, $k \in \{-2^n+1, \ldots, -2^{n-1}-1, 2^{n-1}\} \cup \{2^{n-1}, 2^{n-1}+1, \ldots, 2^n-1\}$. $\square$

Chapter 4

Classical Trigonometric Fourier Series

In this chapter we consider the **approximation of 2π-periodic functions** by a **classical trigonometric Fourier series**. This is a very important and classical approximation technique, and we have already encountered the approximation by trigonometric polynomials briefly as an example in the last chapter. The approximation of functions by a classical trigonometric Fourier series is still widely used in industrial applications.

To analyse the approximation of 2π-periodic functions by classical trigonometric Fourier series, we will draw on all the concepts that we have learned in the last chapter, such as, orthogonal projection, best approximation, Fourier series, and an orthonormal basis. To use these concepts in a concrete context will give them more meaning and will hopefully help you to get a better understanding of them.

A **2π-periodic function** is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$. Such a function is uniquely determined by its values on the interval $[-\pi, \pi]$ with the end points $-\pi$ and π identified (as the considered functions are 2π-periodic). To distinguish $[-\pi, \pi]$ **with the endpoints identified** from the normal interval $[-\pi, \pi]$, we will write $\mathbb{T}$ for $[-\pi, \pi]$ with the endpoints identified.

From a topological point of view, identifying the endpoints of $[-\pi, \pi]$ provides a circle, and we can in fact interpret $\mathbb{T}$ as a parameterisation of the **unit circle** $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 1\}$ via $\exp(i\phi)$, $\phi \in \mathbb{T}$. Thus it is quite natural to use complex trigonometric basis polynomials $e_k(x) = (\sqrt{2\pi})^{-1} \exp(ikx)$, $k \in \mathbb{Z}$, (or equivalently the real trigonometric basis polynomials $\sin(kx)$, $\cos(kx)$, $k \in \mathbb{N}_0$) to approximate functions that are 2π-periodic.

Sets of 2π-periodic functions: We denote the set of 2π-periodic functions on $[-\pi, \pi]$ that are continuous by $C(\mathbb{T})$. The space $C(\mathbb{T})$ of 2π-periodic continuous functions on $\mathbb{T}$ is equipped with the supremum norm

$$\|f\|_{C([-\pi,\pi])} = \sup_{x\in[-\pi,\pi]} |f(x)|, \qquad f \in C(\mathbb{T}). \tag{4.0.1}$$

The set of 2π-periodic functions f on $[-\pi, \pi]$ that are square-integrable, that is, for which

$\|f\|_{L_2([-\pi,\pi])} < \infty$, is denoted by $L_2(\mathbb{T})$. $L_2(\mathbb{T})$ is equipped with the usual $L_2([-\pi,\pi])$ norm

$$\|f\|_{L_2([-\pi,\pi])} = \left(\int_{-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x \right)^{1/2}, \qquad f \in L_2(\mathbb{T}), \tag{4.0.2}$$

and the $L_2([-\pi,\pi])$ inner product

$$\langle f, g \rangle_{L_2([-\pi,\pi])} = \int_{-\pi}^{\pi} f(x) \, \overline{g(x)} \, \mathrm{d}x. \tag{4.0.3}$$

The set of of 2π-periodic functions on $[-\pi,\pi]$ that are k-times continuously differentiable is denoted by $C^k(\mathbb{T})$. Here we mean that $f \in C^k(\mathbb{T})$ is k-times continuously differentiable on $[-\pi,\pi]$ in the sense that $f \in C^k((-\pi,\pi))$ and that at $x = -\pi$ the right-sided derivatives up to order k and at $x = \pi$ the left-sided derivatives up to order k exist and are finite and that $f^{(\ell)}$ is a continuous function on $[-\pi,\pi]$ for $\ell = 0, 1, 2, \ldots, k$.

In this chapter we will consider one particular $L_2([-\pi,\pi])$**-orthonormal basis** for $L_2(\mathbb{T})$ that we have already encountered in the previous chapter: the **complex trigonometric basis polynomials**

$$e_k(x) := \frac{1}{\sqrt{2\pi}} \exp(ikx), \qquad k \in \mathbb{Z},$$

(see Examples 3.19, 3.56, 3.65, and 3.68). Once we have verified that these functions form an $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$, we know from the previous chapter that every 2π-periodic function in $L_2(\mathbb{T})$ can be represented by a **Fourier series** with respect to the complex trigonometric basis polynomials. To show that the complex trigonometric basis polynomials form an $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$ is the main aim of this chapter. The non-trivial proof will be given with the help of **Fejér's theorem**.

Studying classical trigonometric Fourier series and the discrete Fourier transform in this chapter will prepare us for studying the **(discrete) wavelet transform** in the later chapters. As we will see later-on, wavelets provide features that are not given by the classical trigonometric Fourier series; they provide a step beyond the capabilities of classical trigonometric Fourier series analysis, and have been developed over the last 30 years. In contrast, Fourier series are a classical tool that goes back over 200 years.

Remark 4.1 (difference between $L_2([-\pi,\pi])$ and $L_2(\mathbb{T})$)
The distinction between the spaces $L_2([-\pi,\pi])$ and $L_2(\mathbb{T})$ is an 'artificial' one, since these spaces are identical from the Lebesgue integral point of view. Indeed, two functions in $L_2([-\pi,\pi])$ are equivalent if they have the same values apart from sets of Lebesgue measure zero. ('Equivalent' means here that in the topology of $L_2([-\pi,\pi])$ these functions can be identified. More precisely f and g in $L_2([-\pi,\pi])$ are equivalent, if $\|f - g\|_{L_2([-\pi,\pi])} = 0$.) The set $\{\pi\}$ has Lebesgue measure zero, and therefore any function $f \in L_2([-\pi,\pi])$ can be identified with the function $g \in L_2(\mathbb{T})$ defined by $g(x) := f(x)$ for $x \in [-\pi,\pi)$ and $g(\pi) := f(-\pi)$.

4.1 Fejér's Theorem: Complex Trigonometric Basis Polynomials are an $L_2([-\pi,\pi])$-Orthonormal Basis for $L_2(\mathbb{T})$

Our aim is to show that the **family of complex trigonometric basis polynomials**

$$\left\{ e_k(x) = \frac{1}{\sqrt{2\pi}}\, e^{ikx} \,:\, k \in \mathbb{Z} \right\} \tag{4.1.1}$$

is an $L_2([-\pi,\pi])$**-orthonormal basis for** $L_2(\mathbb{T})$ endowed with the $L_2([-\pi,\pi])$ norm (4.0.2) and the $L_2([-\pi,\pi])$ inner product (4.0.3).

From the results in the previous chapter, we know already (see Examples 3.19, 3.56, 3.65, and 3.68) that $\{e_k \,:\, k \in \mathbb{Z}\}$ is an $L_2([-\pi,\pi])$-orthonormal set in $L_2(\mathbb{T})$. To show that $\{e_k \,:\, k \in \mathbb{Z}\}$ is an $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$, we have to show that $\operatorname{span}\{e_k \,:\, k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{T})$, that is, we have to show that for every $f \in L_2(\mathbb{T})$ and every $\epsilon > 0$, there exists an $g \in \operatorname{span}\{e_k \,:\, k \in \mathbb{Z}\}$ such that

$$\|f-g\|_{L_2([-\pi,\pi])} = \left(\int_{-\pi}^{\pi} |f(x)-g(x)|^2 \,\mathrm{d}x \right)^{1/2} < \epsilon.$$

We will achieve this in two steps:

(i) First, we will show that every continuous function $f \in C(\mathbb{T})$ can be approximated arbitrarily well by a functions from $\operatorname{span}\{e_k \,:\, k \in \mathbb{Z}\}$ with respect to the supremum norm (4.0.1). Since the supremum norm (4.0.1) is stronger then the $L_2([-\pi,\pi])$ norm (4.0.2), this implies that we can also approximate $f \in C(\mathbb{T})$ arbitrarily well by functions from $\operatorname{span}\{e_k \,:\, k \in \mathbb{Z}\}$ with respect to the $L_2([-\pi,\pi])$ norm.

(ii) Then, we will use the fact that $C(\mathbb{T})$ is dense in $L_2(\mathbb{T})$ endowed with the $L_2([-\pi,\pi])$ norm (4.0.2). This immediately implies that $\operatorname{span}\{e_k \,:\, k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{T})$ endowed with the $L_2([-\pi,\pi])$ norm (4.0.2). Thus every function in $L_2(\mathbb{T})$ can be approximated by a Fourier series with respect to the $L_2([-\pi,\pi])$-orthonormal set of complex trigonometric basis polynomials $\{e_k \,:\, k \in \mathbb{Z}\}$. Hence the set of complex trigonometric basis polynomials $\{e_k \,:\, k \in \mathbb{Z}\}$ is an $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$.

To show that the complex trigonometric polynomials can be used to approximate functions in $C(\mathbb{T})$ arbitrarily well with respect to the supremum norm (4.0.1), we have to prove the following: Given an arbitrary function $f \in C(\mathbb{T})$, we have to find a sequence $(g_n)_{n\in\mathbb{N}}$ of functions in $\operatorname{span}\{e_k \,:\, k \in \mathbb{Z}\}$ that converges uniformly on $[-\pi,\pi]$ (that is, with respect to the supremum norm (4.0.1)) to f. In formulas, $\lim_{n\to\infty} \|f-g_n\|_{C([-\pi,\pi])} = 0$.

Once we have verified that $\{e_k \,:\, k \in \mathbb{Z}\}$ forms an $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$ with the $L_2([-\pi,\pi])$ norm, we have for any $f \in L_2(\mathbb{T})$

$$f(x) = \sum_{k=-n}^{\infty} \langle f, e_k \rangle_{L_2([-\pi,\pi])}\, e_k(x) = \frac{1}{2\pi} \sum_{k=-n}^{\infty} \left(\int_{-\pi}^{\pi} f(y)\, e^{-iky} \,\mathrm{d}y \right) e^{ikx}, \tag{4.1.2}$$

where the equality holds in the $L_2([-\pi,\pi])$ sense and for almost all $x \in \mathbb{T}$. In particular, this implies that the sequence $(S_n f)_{n\in\mathbb{N}_0}$ of **partial sums** of the complex trigonometric Fourier series (4.1.2)

$$S_n f(x) := \sum_{k=-n}^{n} \langle f, e_k\rangle_{L_2([-\pi,\pi])}\, e_k(x) = \frac{1}{2\pi}\sum_{k=-n}^{n}\left(\int_{-\pi}^{\pi} f(y)\, e^{-iky}\,\mathrm{d}y\right) e^{ikx}, \qquad n \in \mathbb{N}_0. \quad (4.1.3)$$

converges with respect to the $L_2([-\pi,\pi])$ norm to f.

Therefore it would be natural to investigate whether the sequence $(S_n f)_{n\in\mathbb{N}_0}$ of partial sums (4.1.3) also converges uniformly on $[-\pi,\pi]$ (that is, with respect to the supremum norm (4.0.1)) to the function f. However, unfortunately the sequence $(S_n f)_{n\in\mathbb{N}_0}$ does **not** necessarily converge uniformly on $[-\pi,\pi]$ to f, and therefore we need to consider another sequence of linear combinations of the complex trigonometric basis polynomials.

For $f \in C(\mathbb{T})$, consider the linear combination in $U_m = \operatorname{span}\{e_k : k = -m,\ldots,m\}$ defined by

$$F_m(x) := \frac{1}{m+1}\sum_{n=0}^{m} S_n f(x), \qquad m \in \mathbb{N}_0, \quad (4.1.4)$$

where the $S_n f$ are defined by (4.1.3). The function F_m is also referred to as a **Cesaro mean**, since it has the nature of a weighted mean in the following sense: Replacing in (4.1.4) the $S_n f$ by their definition (4.1.3) yields

$$\begin{aligned} F_m(x) &= \frac{1}{m+1}\sum_{n=0}^{m} S_n f(x) \\ &= \frac{1}{m+1}\sum_{n=0}^{m}\sum_{k=-n}^{n}\langle f, e_k\rangle_{L_2([-\pi,\pi])}\, e_k(x) \\ &= \sum_{k=-m}^{m}\frac{m+1-|k|}{m+1}\langle f, e_k\rangle_{L_2([-\pi,\pi])}\, e_k(x), \end{aligned}$$

and we see that the weighting factor $(m+1-|k|)/(m+1)$ declines with increasing $|k|$. However, for any fixed $k \in \mathbb{Z}$, if we increase m, we find that

$$\lim_{m\to\infty}\frac{m+1-|k|}{m+1} = 1,$$

that is, intuitively we expect the series $(F_m)_{m\in\mathbb{N}_0}$ to have the same $L_2(\mathbb{T})$ limit as $(S_n f)_{n\in\mathbb{N}_0}$.

To be able to better manipulate and investigate the functions F_m, $m \in \mathbb{N}_0$, we derive a different representation of these functions as a **convolution with the Fejér kernel**. Interchanging the sum and integral in (4.1.3) yields

$$S_n f(x) = \frac{1}{2\pi}\sum_{k=-n}^{n}\left(\int_{-\pi}^{\pi} f(y)\, e^{-iky}\,\mathrm{d}y\right) e^{ikx}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-n}^{n} f(y)\, e^{-iky}\, e^{ikx}\, \mathrm{d}y$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{k=-n}^{n} e^{ik(x-y)} \right) \mathrm{d}y.$$

Substituting this expression into the definition (4.1.4) of F_m gives

$$\begin{aligned} F_m(x) &= \frac{1}{m+1} \sum_{n=0}^{m} S_n f(x) \\ &= \frac{1}{m+1} \sum_{k=0}^{m} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \left(\sum_{k=-n}^{n} e^{ik(x-y)} \right) \mathrm{d}y \\ &= \int_{-\pi}^{\pi} f(y) \left(\frac{1}{2\pi\,(m+1)} \sum_{n=0}^{m} \sum_{k=-n}^{n} e^{ik(x-y)} \right) \mathrm{d}y, \end{aligned} \tag{4.1.5}$$

where we have interchanged the sum and the integral. Hence, we introduce the **Fejér kernel**.

Definition 4.2 (Fejér kernel)
*For $m \in \mathbb{N}_0$, the **Fejér kernel** $K_m : \mathbb{R} \to \mathbb{C}$ is defined by*

$$K_m(t) := \frac{1}{2\pi\,(m+1)} \sum_{n=0}^{m} \sum_{k=-n}^{n} e^{ikt}, \qquad t \in \mathbb{R}. \tag{4.1.6}$$

For $m = 1, 5, 10$ the Fejér kernel is shown in Figure 4.1.

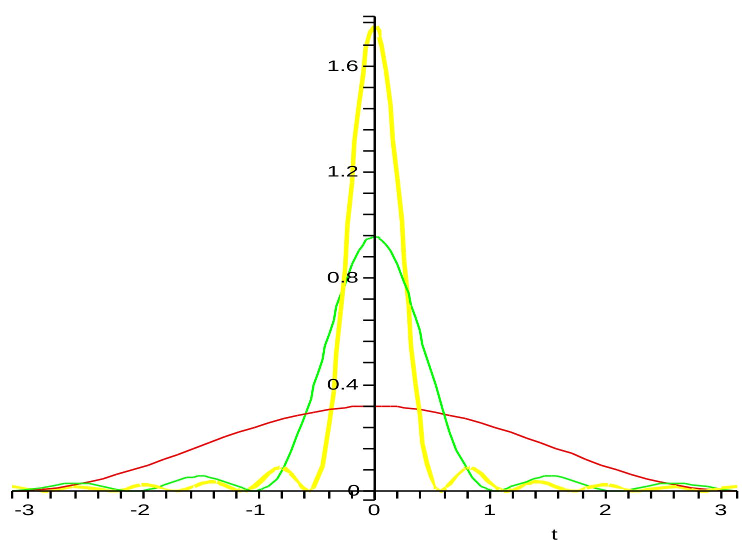

Figure 4.1: Fejér's kernel K_m for $m = 1$ (red), $m = 5$ (green) and $m = 10$ (yellow).

With the Fejér kernel, (4.1.5) has the representation

$$F_m(x) = \int_{-\pi}^{\pi} f(y)\, K_m(x-y)\, \mathrm{d}y.$$

The function F_m is the **convolution** of f and K_m. For $m \to \infty$, the Fejér kernel K_m converges to the delta distribution δ_x, given by

$$\delta_x(f) = \int_{-\pi}^{\pi} f(y)\, \delta_x(y)\, \mathrm{d}y := f(x).$$

More precisely, we have for any $f \in C(\mathbb{T})$ and any $x \in [-\pi,\pi]$ that

$$\delta_x(f) = f(x) = \lim_{m\to\infty} F_m(x) = \lim_{m\to\infty} \int_{-\pi}^{\pi} f(y)\, K_m(x-y)\, \mathrm{d}y. \tag{4.1.7}$$

That (4.1.7) holds is not obvious and follows as a special case from Fejér's theorem below.

Before we state Fejér's theorem, we state some elementary properties of the Fejér kernel.

Lemma 4.3 (elementary properties of the Fejér kernel)
Let $m \in \mathbb{N}_0$. The Fejér kernel $K_m : \mathbb{R} \to \mathbb{C}$, defined by (4.1.6), has the following properties:

(i) $K_m(0) = (m+1)/(2\pi)$.

(ii) K_m *is real-valued.*

(iii) K_m *has the representation*

$$K_m(t) = \frac{1}{2\pi\,(m+1)} \left(m+1+2\sum_{n=0}^{m}\sum_{k=1}^{n} \cos(kt) \right), \qquad t \in \mathbb{R}. \tag{4.1.8}$$

(iv) $K_m(t) \leq K_m(0)$ *for all* $t \in \mathbb{R}$.

The proof of this lemma is left as an exercise.

Exercise 50 *Prove Lemma 4.3.*

From (4.1.8) it seems also plausible that the Fejér kernel has small values for $t \in [-\pi,\pi]$ sufficiently far away from $t = 0$.

Our first aim of this chapter is to prove Fejér's theorem below.

Theorem 4.4 (Fejér's theorem)
For $m \in \mathbb{N}_0$, let $K_m : \mathbb{R} \to \mathbb{C}$ denote the Fejér kernel, defined by (4.1.6). Let $f \in C(\mathbb{T})$. Then the sequence of complex trigonometric polynomials $(F_m)_{m\in\mathbb{N}_0}$, defined by

$$F_m(x) = \int_{-\pi}^{\pi} f(y)\, K_m(x-y)\, \mathrm{d}y, \tag{4.1.9}$$

converges uniformly on $[-\pi,\pi]$ to f, that is,

$$\lim_{m\to\infty} \|F_m - f\|_{C([-\pi,\pi])} = \lim_{m\to\infty} \left(\sup_{x\in[-\pi,\pi]} |F_m(x) - f(x)| \right) = 0.$$

The reader is reminded that the functions F_m in Theorem 4.4 are **complex trigonometric polynomials of degree** m.

Before proving Theorem 4.4 we establish some deeper properties of the Fejér kernel in the next section. These properties are needed in the proof of Fejér's theorem.

Exercise 51 *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C([-\pi,\pi])$ that converges uniformly to a function $g \in C([-\pi,\pi])$, that is,*

$$\lim_{n\to\infty} \|g_n - g\|_{C([-\pi,\pi])} = \lim_{n\to\infty} \left(\sup_{x\in[-\pi,\pi]} |g_n(x) - g(x)| \right) = 0.$$

Show that the sequence $(g_n)_{n\in\mathbb{N}}$ also satisfies $\lim_{n\to\infty} g_n(x) = g(x)$ for all $x \in [-\pi,\pi]$.

Exercise 52 *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of functions in $C([-\pi,\pi])$ that converges uniformly to a function $g \in C([-\pi,\pi])$, that is,*

$$\lim_{n\to\infty} \|g_n - g\|_{C([-\pi,\pi])} = \lim_{n\to\infty} \left(\sup_{x\in[-\pi,\pi]} |g_n(x) - g(x)| \right) = 0.$$

Show that $(g_n)_{n\in\mathbb{N}}$ also converges to g in the $L_2([-\pi,\pi])$ sense, that is,

$$\lim_{n\to\infty} \|g_n - g\|_{L_2([-\pi,\pi])} = \lim_{n\to\infty} \left(\int_{-\pi}^{\pi} |g_n(x) - g(x)|^2 \,\mathrm{d}x \right)^{1/2} = 0.$$

Exercise 53 *Use Fejér's theorem to conclude the following statement:*

$$\overline{\operatorname{span}\left\{ e_k(x) = e^{ikx} \,:\, k \in \mathbb{Z} \right\}}^{\|\cdot\|_{C([-\pi,\pi])}} = C(\mathbb{T}),$$

that is, the span of the complex trigonometric basis polynomials is dense in $C(\mathbb{T})$ endowed with the supremum norm $\|f\|_{C([-\pi,\pi])} := \sup_{x\in[-\pi,\pi]} |f(x)|$.

4.2 Properties of the Fejér Kernel

Our first lemma provides a closed form representation of the Fejér kernel.

Lemma 4.5 (closed form representation for the Fejér kernel)
The Fejer kernel $K_m(t)$, defined by (4.1.6), has the representation

$$K_m(t) = \frac{1}{2\pi(m+1)} \frac{\left[\sin\left(\frac{(m+1)t}{2}\right)\right]^2}{\left[\sin(t/2)\right]^2}, \qquad t \in \mathbb{R}, \tag{4.2.1}$$

where the equality (4.2.1) is pointwise at all $t \in \mathbb{R}$.

Proof of Lemma 4.5: We have to distinguish two separate cases: (i) $t = \ell\,\pi$ with $\ell \in \mathbb{Z}$, and (ii) $t \neq \ell\,\pi$ for all $\ell \in \mathbb{Z}$.

Case (i): Let $t = \ell\,\pi$ with $\ell \in \mathbb{Z}$. Then

$$e^{i\ell\pi} = \begin{cases} 1 = e^{i0} & \text{if} \quad \ell \text{ is even}, \\ -1 = e^{i\pi} & \text{if} \quad \ell \text{ is odd}. \end{cases}$$

Thus we see that $K_m(t)$, defined by (4.1.6), satisfies

$$K_m(\ell\,\pi) = \begin{cases} K_m(0) = \dfrac{(m+1)}{2\pi} & \text{if} \quad \ell \text{ is even}, \\ K_m(\pi) = \dfrac{1}{2\pi\,(m+1)} & \text{if} \quad \ell \text{ is odd and } m \text{ is even}, \\ K_m(\pi) = 0 & \text{if} \quad \ell \text{ is odd and } m \text{ is odd}, \end{cases}$$

where $K_m(0) = (m+1)/(2\pi)$ is Lemma 4.3 (i), and where the formula for $K_m(\pi)$ follows from $e^{i\pi} = -1$ and

$$K_m(\pi) = \frac{1}{2\pi\,(m+1)} \sum_{n=0}^{m} \sum_{k=-n}^{n} (-1)^k = \frac{1}{2\pi\,(m+1)} \sum_{n=0}^{m} (-1)^n = \begin{cases} \dfrac{1}{2\pi\,(m+1)} & \text{if} \quad m \text{ is even}, \\ 0 & \text{if} \quad m \text{ is odd}. \end{cases}$$

By evaluating the right-hand side in (4.2.1) for $t = 0$ and $t = \ell\pi$, $\ell \in \mathbb{Z}$, explicitly, it can be shown that (4.2.1) indeed holds for $t = 0$ and $t = \ell\pi$ with $\ell \in \mathbb{Z}$. This is left as an exercise.

Case (ii): Now let $z = e^{it}$ with $t \neq \ell\pi$ where $\ell \in \mathbb{Z}$. For brevity, set $z = e^{it}$. Then we have

$$2\pi\,(m+1)\,K_m(t) = \sum_{n=0}^{m} \sum_{k=-n}^{n} z^k. \tag{4.2.2}$$

As $t \in \mathbb{R}$, we then find $\bar{z} = e^{-it} = (e^{it})^{-1} = z^{-1}$. Thus using $z^{-1} = \bar{z}$ and $\bar{z}\,z = 1$ and the formula for the geometric sum, we can transform the inner sum as follows

$$\sum_{k=-n}^{n} z^k = \bar{z}^n \sum_{k=-n}^{n} z^n\,z^k = \bar{z}^n \sum_{k=-n}^{n} z^{n+k} = \bar{z}^n \sum_{k=0}^{2n} z^k = \bar{z}^n \frac{1 - z^{2n+1}}{1-z} = \frac{\bar{z}^n - z^{n+1}}{1-z}. \tag{4.2.3}$$

Therefore, substituting (4.2.3) into (4.2.2) and using again the geometric sum and $\bar{z}\,z = 1$ and $\bar{z} = z^{-1}$ and finally the binomial formula $a^2 - 2\,a\,b + b^2 = (a-b)^2$,

$$\begin{aligned} 2\pi\,(m+1)\,K_m(t) &= \sum_{n=0}^{m} \frac{\bar{z}^n - z^{n+1}}{1-z} \\ &= \frac{1}{1-z} \left(\sum_{n=0}^{m} \bar{z}^n - z \sum_{n=0}^{m} z^n \right) \\ &= \frac{1}{1-z} \left(\frac{1 - \bar{z}^{m+1}}{1-\bar{z}} - \frac{z\,(1 - z^{m+1})}{1-z} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-z}\left(\frac{1-\bar{z}^{m+1}}{1-\bar{z}}+\frac{1-z^{m+1}}{1-\bar{z}}\right)\\
&= \frac{1}{|1-z|^2}\left(2-\bar{z}^{m+1}-z^{m+1}\right)\\
&= -\frac{z^{m+1}-2+z^{-m-1}}{|1-z|^2}\\
&= -\frac{\left(z^{\frac{m+1}{2}}\right)^2-2+\left(z^{-\frac{m+1}{2}}\right)^2}{|1-z|^2}\\
&= -\frac{\left(z^{\frac{m+1}{2}}-z^{-\frac{m+1}{2}}\right)^2}{|1-z|^2}. \qquad (4.2.4)
\end{aligned}$$

Now, using $|e^{it/2}|=1$ and Euler's formula, the denominator is given by

$$|1-z|^2 = |1-e^{it}|^2 = |e^{it/2}\left(e^{-it/2}-e^{it/2}\right)|^2 = |e^{it/2}-e^{-it/2}|^2 = \big(2|\sin(t/2)|\big)^2 = \big(2\sin(t/2)\big)^2. \qquad (4.2.5)$$

Moreover, using $z^{-1}=\bar{z}$ and $\bar{z}^\ell=\overline{z^\ell}$ for $z=e^{it}$ and Euler's formula, the numerator equals

$$\left(z^{\frac{m+1}{2}}-z^{-\frac{m+1}{2}}\right)^2 = \left(e^{i\frac{m+1}{2}t}-e^{-i\frac{m+1}{2}t}\right)^2 = \left[2i\,\sin\left(\frac{(m+1)t}{2}\right)\right]^2 = -\left[2\,\sin\left(\frac{(m+1)t}{2}\right)\right]^2. \qquad (4.2.6)$$

Now the formula (4.2.1) follows from substituting (4.2.5) and (4.2.6) into (4.2.4). □

Lemma 4.6 (properties of the Fejér kernel)
The Fejér kernel possesses the following properties:

(i) The Fejér kernel is 2π-periodic, that is, $K_m(t)=K_m(t+2\pi)$ for any $t\in\mathbb{R}$.

(ii) The Fejér kernel is an even function, that is, $K_m(t)=K_m(-t)$ for all $t\in\mathbb{R}$.

(iii) $K_m(t)\geq 0$ for all $t\in\mathbb{R}$.

(iv) The Fejér kernel is normalised such that

$$\int_{-\pi}^{\pi} K_m(t)\,\mathrm{d}t = 1. \qquad (4.2.7)$$

(v) For any $\delta\in(0,\pi)$

$$\int_{t\in[-\pi,\pi]\setminus(-\delta,\delta)} K_m(t)\,\mathrm{d}t \to 0 \qquad \text{as } m\to\infty. \qquad (4.2.8)$$

Proof of Lemma 4.6:

(i) The 2π-periodicity of K_m follows directly from the formula (4.1.6) of the Fejér kernel K_m.

(ii) That K_m is an even function follows from the closed form representation (4.2.1) of K_m and from $\sin(-t) = -\sin(t)$ for all $t \in \mathbb{R}$. Indeed,

$$\begin{aligned} K_m(-t) &= \frac{1}{2\pi(m+1)} \frac{\left[\sin\left(\frac{(m+1)(-t)}{2}\right)\right]^2}{\left[\sin(-t/2)\right]^2} \\ &= \frac{1}{2\pi(m+1)} \frac{\left[-\sin\left(\frac{(m+1)t}{2}\right)\right]^2}{\left[-\sin(t/2)\right]^2} \\ &= \frac{1}{2\pi(m+1)} \frac{\left[\sin\left(\frac{(m+1)t}{2}\right)\right]^2}{\left[\sin(t/2)\right]^2} \\ &= K_m(t) \qquad \text{for all } t \in \mathbb{R}. \end{aligned}$$

(iii) $K_m(t) \geq 0$ for all $t \in \mathbb{R}$ follows directly from the closed form representation (4.2.1) in Lemma 4.5.

(iv) The normalization (4.2.7) in property (iv) follows from (4.1.6) and the equality

$$\int_{-\pi}^{\pi} e^{ikt} dt = \begin{cases} 2\pi & \text{if } k = 0, \\ 0 & \text{if } k \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Indeed, using the last formula and the original definition (4.1.6) of K_m,

$$\begin{aligned} \int_{-\pi}^{\pi} K_m(t)\, \mathrm{d}t &= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi\,(m+1)} \sum_{n=0}^{m} \sum_{k=-n}^{n} e^{ikt} \right) \mathrm{d}t \\ &= \frac{1}{2\pi\,(m+1)} \sum_{n=0}^{m} \sum_{k=-n}^{n} \int_{-\pi}^{\pi} e^{ikt}\, \mathrm{d}t \\ &= \frac{1}{2\pi\,(m+1)} \sum_{n=0}^{m} 2\pi \\ &= \frac{2\pi\,(m+1)}{2\pi\,(m+1)} \;=\; 1. \end{aligned}$$

(v) To prove the limit (4.2.8) in property (v), notice that we have

$$\left[\sin(t/2)\right]^2 \geq \left[\sin(\delta/2)\right]^2 \qquad \text{for all } t \in [-\pi, \pi] \setminus (-\delta, \delta).$$

Therefore, using the closed form representation (4.2.1) of K_m and $[\sin(\tau)]^2 \leq 1$ for all $\tau \in \mathbb{R}$,

$$K_m(t) = \frac{1}{2\pi(m+1)} \frac{\left[\sin\left(\frac{(m+1)t}{2}\right)\right]^2}{\left[\sin(t/2)\right]^2} \leq \frac{1}{2\pi\,(m+1)} \frac{1}{\left[\sin(\delta/2)\right]^2} \qquad \text{for all } t \in [-\pi, \pi] \setminus (-\delta, \delta).$$

Thus

$$\int_{t \in [-\pi,\pi] \setminus (-\delta,\delta)} K_m(t)\, \mathrm{d}t \;\leq\; \frac{1}{2\pi\,(m+1)} \int_{t \in [-\pi,\pi] \setminus (-\delta,\delta)} \frac{1}{\left[\sin(\delta/2)\right]^2}\, \mathrm{d}t$$

$$= \frac{1}{2\pi\,(m+1)}\,\frac{2\pi - 2\delta}{\big[\sin(\delta/2)\big]^2} \;\to\; 0 \qquad \text{as } m \to \infty,$$

which proves (4.2.8).
This completes the proof. □

Note that from the 2π-periodicity of the Fejér kernel, we see immediately

$$\int_{-\pi}^{\pi} K_m(t+a)\,\mathrm{d}t = 1 \qquad \text{for all } a \in \mathbb{R}. \tag{4.2.9}$$

Exercise 54 *Verify that the right-hand side of formula (4.2.1) satisfies*

$$\frac{1}{2\pi(m+1)}\,\frac{\Big[\sin\Big(\frac{(m+1)\ell\pi}{2}\Big)\Big]^2}{\big[\sin(\ell\pi/2)\big]^2} = \begin{cases} \dfrac{(m+1)}{2\pi} & \text{if } \ \ell \text{ is even,} \\ \dfrac{1}{2\pi\,(m+1)} & \text{if } \ \ell \text{ is odd and } m \text{ is even,} \\ 0 & \text{if } \ \ell \text{ is odd and } m \text{ is odd.} \end{cases}$$

4.3 Proof of Fejér's Theorem

After the preparations in the last section, we can now prove Fejér's theorem.

Proof of Theorem 4.4: Let $f \in C(\mathbb{T})$ be an arbitrary function. We want to prove that for any $\epsilon > 0$ there exists an integer $M = M(\epsilon) \in \mathbb{N}$ such that

$$\|f - F_m\|_{C([-\pi,\pi])} = \sup_{x\in[-\pi,\pi]} |f(x) - F_m(x)| < \epsilon \qquad \text{for all } m \geq M. \tag{4.3.1}$$

This then implies that

$$\lim_{m\to\infty} \|f - F_m\|_{C([-\pi,\pi])} = \lim_{m\to\infty} \left(\sup_{x\in[-\pi,\pi]} |f(x) - F_m(x)| \right) = 0,$$

that is, $(F_m)_{m\in\mathbb{N}}$ converges uniformly on $[-\pi,\pi]$ to f.

First we write $f(x) - F_m(x)$ in a more convenient form. Using the equation (4.2.9) and the 2π-periodicity of K_m (see Lemma 4.6 (i)) and the 2π-periodicity of f, we can write

$$\begin{aligned} f(x) - F_m(x) &= f(x) - \int_{-\pi}^{\pi} f(y)\,K_m(x-y)\,\mathrm{d}y \\ &= \int_{-\pi}^{\pi} f(x)\,K(x-y)\,\mathrm{d}y - \int_{-\pi}^{\pi} f(y)\,K_m(x-y)\,\mathrm{d}y \\ &= \int_{-\pi}^{\pi} \big[f(x) - f(y)\big]\,K_m(x-y)\,\mathrm{d}y \\ &= \int_{x-\pi}^{x+\pi} \big[f(x) - f(y)\big]\,K_m(x-y)\,\mathrm{d}y. \end{aligned}$$

Let us change the variable with the substitution $y = t + x$, or equivalently, $t = y - x$. Then, using $K_m(t) = K_m(-t)$ for all $t \in \mathbb{R}$ (see Lemma 4.6 (ii)), the previous equality will take the form

$$f(x) - F_m(x) = \int_{-\pi}^{\pi} \big[f(x) - f(x+t)\big]\, K_m(-t)\, \mathrm{d}t = \int_{-\pi}^{\pi} \big[f(x) - f(x+t)\big]\, K_m(t)\, \mathrm{d}t.$$

Now let us fix an arbitrary $\epsilon > 0$. Due to the uniform continuity of $f \in C(\mathbb{T})$, there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\sup_{x\in[-\pi,\pi]} |f(x+t) - f(x)| < \frac{\epsilon}{2} \qquad \text{for all } |t| \le \delta. \tag{4.3.2}$$

(That a continuous function $f \in C([a,b])$ is uniformly continuous on the bounded interval $[a,b]$ will be shown as an exercise.) First we split the integral over $[-\pi,\pi]$ into one integral over $(-\delta,\delta)$ and one integral over $[-\pi,\pi] \setminus (-\delta,\delta)$, yielding

$$f(x) - F_m(x) = \int_{-\delta}^{\delta} \big[f(x) - f(x+t)\big]\, K_m(t)\, \mathrm{d}t + \int_{[-\pi,\pi]\setminus(-\delta,\delta)} \big[f(x) - f(x+t)\big]\, K_m(t)\, \mathrm{d}t.$$

Then, using $K_m(t) \ge 0$ for all $t \in \mathbb{R}$ (see Lemma 4.6 (i)), the uniform continuity (4.3.2) of f, Lemma 4.6 (iv), and the 2π-periodicity of f, we find for any $x \in [-\pi,\pi]$,

$$\begin{aligned}
&|f(x) - F_m(x)| \\
&\le \int_{-\delta}^{\delta} |f(x) - f(x+t)|\, K_m(t)\, \mathrm{d}t + \int_{[-\pi,\pi]\setminus(-\delta,\delta)} |f(x) - f(x+t)|\, K_m(t)\, \mathrm{d}t \\
&\le \int_{-\delta}^{\delta} \frac{\epsilon}{2}\, K_m(t)\, \mathrm{d}t + \int_{[-\pi,\pi]\setminus(-\delta,\delta)} \big(|f(x)| + |f(x+t)|\big)\, K_m(t)\, \mathrm{d}t \\
&\le \frac{\epsilon}{2} \underbrace{\int_{-\pi}^{\pi} K_m(t)\, \mathrm{d}t}_{=1} + \int_{[-\pi,\pi]\setminus(-\delta,\delta)} \Big(2 \sup_{y\in\mathbb{R}} |f(y)|\Big)\, K_m(t)\, \mathrm{d}t \\
&\le \frac{\epsilon}{2} + 2 \underbrace{\Big(\sup_{y\in[-\pi,\pi]} |f(y)|\Big)}_{= \|f\|_{C([-\pi,\pi])}} \int_{[-\pi,\pi]\setminus(-\delta,\delta)} K_m(t)\, \mathrm{d}t.
\end{aligned} \tag{4.3.3}$$

The last integral tends to zero as $m \to \infty$ by Lemma 4.6 (v). Therefore there exists an $M = M(\epsilon)$ such that

$$2\, \|f\|_{C([-\pi,\pi])} \int_{[-\pi,\pi]\setminus(-\delta,\delta)} K_m(t)\, \mathrm{d}t < \frac{\epsilon}{2} \qquad \text{for all } m \ge M. \tag{4.3.4}$$

Thus from (4.3.3) and (4.3.4), given $\epsilon > 0$, there exists an $M = M(\epsilon) \in \mathbb{N}$ such that

$$|f(x) - F_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for all } m \ge M \qquad \text{and for all } x \in [-\pi,\pi].$$

Thus

$$\|f - F_m\|_{C([-\pi,\pi])} = \sup_{x\in[-\pi,\pi]} |f(x) - F_m(x)| < \epsilon \qquad \text{for all } m \ge M,$$

which shows that $(F_m)_{m\in\mathbb{N}}$ converges uniformly on $[-\pi,\pi]$ to f. □

Exercise 55 *Let $f \in C([a,b])$ be a continuous function on $[a,b]$. Show that f is uniformly continuous on $[a,b]$, that is, show that for every $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$|f(x) - f(y)| < \epsilon \qquad \text{for all } x, y \in [a,b] \text{ with } |x-y| < \delta.$$

4.4 Completeness of the Complex Trigonometric Basis Functions

A direct consequence of Fejér's theorem (see Theorem 4.4) is the following corollary.

Corollary 4.7 (trigonometric polynomials are dense in $C(\mathbb{T})$ w.r.t. $\|\cdot\|_{L_2([-\pi,\pi])}$) *Let $f \in C(\mathbb{T})$. Then for every $\epsilon > 0$, there exists a complex trigonometric polynomial $p \in \operatorname{span}\{e_k(x) = (\sqrt{2\pi})^{-1} e^{ikx} \,:\, k \in \mathbb{Z}\}$, such that*

$$\|f - p\|_{L_2([-\pi,\pi])} = \left(\int_{-\pi}^{\pi} |f(x) - p(x)|^2 \, \mathrm{d}x \right)^{1/2} < \epsilon.$$

In other words,

$$C(\mathbb{T}) \subset \overline{\operatorname{span}\left\{ e_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx} \,:\, k \in \mathbb{Z} \right\}}^{\|\cdot\|_{L_2([-\pi,\pi])}}.$$

Proof of Corollary 4.7: Given $f \in C(\mathbb{T})$, let F_m denote the function defined by (4.1.9) in Theorem 4.4. From Theorem 4.4, for every $\epsilon > 0$, there exists an $M = M(\epsilon)$ such that

$$\sup_{x \in [-\pi,\pi]} |f(x) - F_m(x)| < \frac{\epsilon}{\sqrt{2\pi}} \qquad \text{for all } m \geq M. \tag{4.4.1}$$

The function F_m is by definition in $\operatorname{span}\{e_k(x) = (\sqrt{2\pi})^{-1} e^{ikx} \,:\, k = -m, \ldots, m\}$ and hence is a complex trigonometric polynomial of degree $\leq m$. From (4.4.1) with $m = M$

$$\begin{aligned}
\|f - F_M\|_{L_2([-\pi,\pi])}^2 &= \int_{-\pi}^{\pi} |f(x) - F_M(x)|^2 \, \mathrm{d}x \\
&\leq \int_{-\pi}^{\pi} \left(\sup_{y \in [-\pi,\pi]} |f(y) - F_M(y)| \right)^2 \mathrm{d}x \\
&< \int_{-\pi}^{\pi} \frac{\epsilon^2}{2\pi} \, \mathrm{d}x \\
&= \frac{\epsilon^2}{2\pi} 2\pi \;=\; \epsilon^2,
\end{aligned}$$

which proves $\|f - F_M\|_{L_2([-\pi,\pi])} < \epsilon$. □

To show that the span of the complex trigonometric basis polynomials is dense in $L_2(\mathbb{T})$, endowed with the $L_2([-\pi,\pi])$-norm, we have to use the fact that the continuous 2π-periodic functions on $[-\pi,\pi]$ are dense in $L_2(\mathbb{T})$, that is, $C(\mathbb{T})$ is dense in $L_2(\mathbb{T})$.

Theorem 4.8 ($C([a,b])$ is dense in $L_2([a,b])$ with $\|\cdot\|_{L_2([a,b])}$)
The continuous functions on $[a,b]$ are dense in $L_2([a,b])$ with respect to the $L_2([a,b])$ norm. That is, for every $f \in L_2([a,b])$ and for every $\epsilon > 0$ there exists a function $g \in C([a,b])$ such that

$$\|f-g\|_{L_2([a,b])} = \left(\int_a^b |f(x)-g(x)|^2 \, \mathrm{d}t\right)^{1/2} < \epsilon.$$

The proof of Theorem 4.8 demands a deep knowledge of the Lebesgue integral and can therefore not be given in this course.

Corollary 4.7 and Theorem 4.8 allow us finally to show that $\{e_k(x) = (\sqrt{2\pi})^{-1}\, e^{ikx} \,:\, k \in \mathbb{Z}\}$ is an $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$.

Theorem 4.9 ($\{e_k \,:\, k \in \mathbb{Z}\}$ is a $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$)
*The set of complex trigonometric basis polynomials $\{e_k(x) = (\sqrt{2\pi})^{-1}\, e^{ikx} \,:\, k \in \mathbb{Z}\}$ forms an $L_2([-\pi,\pi])$-**orthonormal basis** for $L_2(\mathbb{T})$ endowed with the $L_2([-\pi,\pi])$ norm $\|\cdot\|_{L_2([-\pi,\pi])}$. Any function $f \in L_2(\mathbb{T})$ can be represented in the $L_2(\mathbb{T})$ sense as the **(trigonometric) Fourier series***

$$f(x) = \sum_{k=-\infty}^{\infty} \widehat{f}_k \, e_k(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \widehat{f}_k \, e^{ikx} \tag{4.4.2}$$

*with the **Fourier coefficients***

$$\widehat{f}_k := \langle f, e_k \rangle_{L_2([-\pi,\pi])} = \int_{-\pi}^{\pi} f(x)\, \overline{e_k(x)} \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, e^{-ikx} \, \mathrm{d}x. \tag{4.4.3}$$

Proof of Theorem 4.9: First we note that from Examples 3.19, 3.56, 3.65, and 3.68 in Chapter 3 we already know that the set $\{e_k \,:\, k \in \mathbb{Z}\}$ of complex trigonometric basis polynomials is an $L_2([-\pi,\pi])$-orthonormal set in $L_2(\mathbb{T})$. It remains to show that this $L_2([-\pi,\pi])$-orthonormal set is also an $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$.

The $L_2([-\pi,\pi])$-orthonormal set $\{e_k \,:\, k \in \mathbb{Z}\}$ is an $L_2([-\pi,\pi])$-orthonormal basis for $L_2(\mathbb{T})$, if span $\{e_k \,:\, k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{T})$ endowed with the $L_2(\mathbb{T})$-norm, that is, if for every $f \in L_2(\mathbb{T})$ and for $\epsilon > 0$ there exists a function g in span $\{e_k \,:\, k \in \mathbb{Z}\}$ such that

$$\|f-g\|_{L_2([-\pi,\pi])} < \epsilon.$$

Now let f in $L_2(\mathbb{T})$ be arbitrary, and let $\epsilon > 0$ be arbitrary. Then from Theorem 4.8 with $[a,b] = [-\pi,\pi]$ there exists $h \in C([-\pi,\pi])$ such that

$$\|f-h\|_{L_2([-\pi,\pi])} < \frac{\epsilon}{2}. \tag{4.4.4}$$

It is reasonably easily verified that $h \in C([-\pi,\pi])$ can in fact be chosen 2π-periodic, such that in (4.4.4) we have $h \in C(\mathbb{T})$. For this $h \in C(\mathbb{T})$, from Corollary 4.7, there exists a function $g \in \operatorname{span}\{e_k \,:\, k \in \mathbb{Z}\}$ such that

$$\|h-g\|_{L_2([-\pi,\pi])} < \frac{\epsilon}{2}. \tag{4.4.5}$$

Hence, from the triangle inequality and (4.4.4) and (4.4.5),

$$\|f-g\|_{L_2([-\pi,\pi])} \leq \|f-h\|_{L_2([-\pi,\pi])} + \|h-g\|_{L_2([-\pi,\pi])} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves that $\operatorname{span}\{e_k \,:\, k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{T})$, endowed with the $L_2([-\pi,\pi])$ norm.

The formulas (4.4.2) and (4.4.3) follow now directly from Theorem 3.62. □

Corollary 4.10 (inner product in terms of the Fourier coefficients)
Let $f, g \in L_2(\mathbb{T})$. Then

$$\langle f, g\rangle_{L_2([-\pi,\pi])} = \sum_{k=-\infty}^{\infty} \widehat{f}_k\, \overline{\widehat{g}_k}.$$

Proof of Corollary 4.10: Using (4.4.2) for f and g, the sesqui-linearity of the inner product, the fact that the inner product $\langle \cdot,\cdot\rangle_{L_2([-\pi,\pi])}$ is continuous, and the $L_2([-\pi,\pi])$-orthonormality $\langle e_k, e_\ell\rangle_{L_2([-\pi,\pi])} = \delta_{k,\ell}$ of the complex trigonometric basis functions e_k, we can write:

$$\begin{aligned}
\langle f, g\rangle_{L_2([-\pi,\pi])} &= \left\langle \sum_{k=-\infty}^{\infty} \widehat{f}_k\, e_k, \sum_{\ell=-\infty}^{\infty} \widehat{g}_k\, e_k \right\rangle_{L_2([-\pi,\pi])} \\
&= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \widehat{f}_k\, \overline{\widehat{g}_\ell}\, \underbrace{\langle e_k, e_\ell\rangle_{L_2([-\pi,\pi])}}_{=\delta_{k,\ell}} \\
&= \sum_{k=-\infty}^{\infty} \widehat{f}_k\, \overline{\widehat{g}_k}.
\end{aligned}$$

In the previous formula, it is allowed to pull the infinite sums out of the inner product because the Fourier series converges in the $L_2([-\pi,\pi])$ sense and because the inner product is continuous (see Lemma 3.10). □

Corollary 4.11 (Fourier series of an ℓ-times continuously differentiable function)
Let f be in the space $C^\ell(\mathbb{T})$ of 2π-periodic ℓ-times continuously differentiable functions on $[-\pi,\pi]$, and assume that the 1st to ℓth derivative are also 2π-periodic. Then the following holds true:

(i) The Fourier coefficients of the ℓth derivative $f^{(\ell)}$ of f are given by

$$(\widehat{f^{(\ell)}})_k = (ik)^\ell\, \widehat{f}_k, \qquad k \in \mathbb{Z}. \tag{4.4.6}$$

(ii) If $\ell \geq 2$, then the Fourier series of f converges uniformly on $[-\pi,\pi]$ to f.

Proof of Corollary 4.11: For the Fourier coefficient $(\widehat{f'})_k$ of the first derivative f' of f we can use integration by parts to derive

$$\begin{aligned}(\widehat{f'})_k &= \langle f', e_k \rangle_{L_2([-\pi,\pi])} \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f'(x)\, e^{-ikx}\, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \big[f(x)\, e^{-ikx} \big]_{-\pi}^{\pi} + ik\, \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, \mathrm{d}x \\ &= ik\, \widehat{f}_k,\end{aligned}$$

where the boundary term vanishes due to the 2π-periodicity of the function f. This verifies

$$(\widehat{f'})_k = ik\, \widehat{f}_k, \qquad k \in \mathbb{Z}, \tag{4.4.7}$$

which is the formula (4.4.6) for $\ell = 1$. The formula (4.4.7) for $\ell = 1$ also verifies for $f'' = (f')'$

$$(\widehat{f''})_k = \big(\widehat{(f')'}\big)_k = ik\, (\widehat{f'})_k = ik\, (ik\, \widehat{f}_k) = (ik)^2\, \widehat{f}_k,$$

where we have applied (4.4.7) twice, once for the derivative of f' and once for the derivative of f itself. This proves (4.4.6) for $\ell = 2$. Iterating this process gives (4.4.6) for any $\ell \in \mathbb{N}$.

For the second statement note that $\ell \geq 2$ implies $(\widehat{f''})_k = -k^2\, \widehat{f}_k$ which implies

$$|\widehat{f}_k| = \frac{|(\widehat{f''})_k|}{k^2}, \qquad k \in \mathbb{Z} \setminus \{0\}. \tag{4.4.8}$$

Next we estimate $(\widehat{f''})_k$, making use of the fact that $f \in C^\ell(\mathbb{T})$ with $\ell \geq 2$ and hence f'' is continuous on $\mathbb{T}$. We find for all $k \in \mathbb{Z}$,

$$\begin{aligned}|(\widehat{f''})_k| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f''(x)\, e^{-ikx}\, \mathrm{d}x \right| \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |f''(x)|\, \underbrace{|e^{-ikx}|}_{=1}\, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |f''(x)|\, \mathrm{d}x \\ &\leq \frac{1}{\sqrt{2\pi}} \left(\sup_{x \in [-\pi,\pi]} |f''(x)| \right) \int_{-\pi}^{\pi} 1\, \mathrm{d}x \\ &= \sqrt{2\pi}\, \|f''\|_{C([-\pi,\pi])}.\end{aligned}$$

Applying the last estimate in (4.4.8), we obtain

$$|\widehat{f}_k| \leq \frac{\sqrt{2\pi}}{k^2}\, \|f''\|_{C([-\pi,\pi])}, \qquad k \in \mathbb{Z} \setminus \{0\}. \tag{4.4.9}$$

With the help of (4.4.9), we can now verify that the Fourier series converges absolutely and uniformly on $[-\pi,\pi]$. Indeed, consider the series of partial sums $(S_n f)_{n\in\mathbb{N}_0}$ of the Fourier series of f, defined by

$$S_n f(x) := \sum_{k=-n}^{n} \widehat{f}_k \, e_k(x), \qquad x \in [-\pi,\pi].$$

Then given $\epsilon > 0$, there exists some $N = N(\epsilon) \in \mathbb{N}$ such that for all $n, m \geq N$, where $n \geq m$,

$$\begin{aligned}
\sup_{x\in[-\pi,\pi]} |S_n f(x) - S_m f(x)| &= \sup_{x\in[-\pi,\pi]} \left| \sum_{k=-n}^{n} \widehat{f}_k \, e_k(x) - \sum_{k=-m}^{m} \widehat{f}_k \, e_k(x) \right| \\
&= \sup_{x\in[-\pi,\pi]} \left| \sum_{\substack{k\in\mathbb{Z},\\ m<|k|\leq n}} \widehat{f}_k \, e_k(x) \right| \\
&\leq \sup_{x\in[-\pi,\pi]} \left(\sum_{\substack{k\in\mathbb{Z},\\ m<|k|\leq n}} |\widehat{f}_k| \, |e_k(x)| \right) \\
&\leq \sup_{x\in[-\pi,\pi]} \left(\sum_{\substack{k\in\mathbb{Z},\\ m<|k|\leq n}} \frac{\sqrt{2\pi}}{k^2} \|f''\|_{C([-\pi,\pi])} \frac{1}{\sqrt{2\pi}} \underbrace{|e^{ikx}|}_{\leq 1} \right) \\
&\leq \|f''\|_{C([-\pi,\pi])} \sum_{\substack{k\in\mathbb{Z},\\ m<|k|\leq n}} \frac{1}{k^2} \\
&\leq \|f''\|_{C([-\pi,\pi])} \, 2 \int_m^n \frac{1}{t^2} \, \mathrm{d}t \\
&= \|f''\|_{C([-\pi,\pi])} \, 2 \left[\frac{(-1)}{t} \right]_m^n \\
&\leq \|f''\|_{C([-\pi,\pi])} \frac{2}{m} \\
&< \epsilon,
\end{aligned}$$

where we have used (4.4.9) and the estimate from the integral test for the convergence of a series. (More precisely, $N = N(\epsilon)$ was chosen to be smallest integer strictly larger then $2\,\|f''\|_{C([-\pi,\pi])}/\epsilon$.) This means that $(S_n f)_{n\in\mathbb{N}_0}$ is a Cauchy sequence in the Banach space $C([-\pi,\pi])$ endowed with the supremum norm $\|g\|_{C([-\pi,\pi])} = \sup_{x\in[-\pi,\pi]} |g(x)|$. As $C([-\pi,\pi])$ is complete, this Cauchy sequence $(S_n f)_{n\in\mathbb{N}_0}$ converges uniformly on $[-\pi,\pi]$ to some continuous function g. As the $S_n f$ are all 2π-periodic, the uniform (and pointwise) limit g is also 2π-periodic, hence $g \in C(\mathbb{T})$. Because convergence in $C([-\pi,\pi])$ implies convergence in $L_2([-\pi,\pi])$, the Cauchy sequence $(S_n f)_{n\in\mathbb{N}_0}$ converges also in $L_2([-\pi,\pi])$ to g. But the $L_2([-\pi,\pi])$ limit is unique, and therefore we know that $g = f$, since $(S_n f)_{n\in\mathbb{N}_0}$ converges in $L_2([-\pi,\pi])$ to f. Hence the sequence of the partial sums $(S_n f)_{n\in\mathbb{N}_0}$ and the Fourier series converge uniformly on $[-\pi,\pi]$ to f. □

Another consequence of the fast decay of the Fourier coefficients of a smooth function is that we can give a **convergence order** and an **error bound** with an explicit constant for the approximation of a smooth functions by the partial sums of its Fourier series.

Corollary 4.12 (error for approximation by the nth partial sum if $f \in C^\ell(\mathbb{T})$) *Let $f \in C^\ell(\mathbb{T})$, where $\ell \geq 2$, and assume that all continuous derivatives $f^{(m)}$ for $m = 1, 2, \ldots, \ell$ are also 2π-periodic. Then, the nth partial sum*

$$S_n f = \sum_{k=-n}^{n} \widehat{f}_k \, e_k$$

is an approximation to f, which satisfies the ***uniform error estimate***

$$\|f - S_n f\|_{C([-\pi,\pi])} = \sup_{x \in [-\pi,\pi]} |f(x) - S_n f(x)| \leq \frac{c}{n^{\ell-1}}, \tag{4.4.10}$$

where the positive constant c is explicitly given by

$$c := \frac{2}{(\ell-1)} \, \|f^{(\ell)}\|_{C([-\pi,\pi])} = \frac{2}{(\ell-1)} \sup_{x \in [-\pi,\pi]} |f^{(\ell)}(x)|.$$

Since the constant can be computed explicitly, this gives us information on how many Fourier coefficients $\widehat{f}_k$, $k = -n, \ldots, n$, we need to compute so that $S_n f$ has a given accuracy.

Proof of Corollary 4.12: Using Corollary 4.11 (i), we know that $\widehat{(f^{(\ell)})}_k = (ik)^\ell \, \widehat{f}_k$, which implies that

$$\widehat{f}_k = \left(\frac{\bar{i}}{k}\right)^\ell \widehat{(f^{(\ell)})}_k \qquad \text{for all } k \in \mathbb{Z} \setminus \{0\}. \tag{4.4.11}$$

Hence we have for any $x \in [-\pi, \pi]$

$$\begin{aligned}
|f(x) - S_n f(x)| &= \left| \sum_{k \in \mathbb{Z}} \widehat{f}_k \, e_k(x) - \sum_{k=-n}^{n} \widehat{f}_k \, e_k(x) \right| \\
&\leq \sum_{\substack{k \in \mathbb{Z}, \\ |k| > n}} |\widehat{f}_k| \, |e_k(x)| \\
&= \sum_{\substack{k \in \mathbb{Z}, \\ |k| > n}} |\widehat{f}_k| \, \frac{1}{\sqrt{2\pi}} \underbrace{|e^{ikx}|}_{\leq 1} \\
&\leq \frac{1}{\sqrt{2\pi}} \sum_{\substack{k \in \mathbb{Z}, \\ |k| > n}} |\widehat{f}_k| \\
&= \frac{1}{\sqrt{2\pi}} \sum_{\substack{k \in \mathbb{Z}, \\ |k| > n}} \frac{|\widehat{(f^{(\ell)})}_k|}{|k|^\ell},
\end{aligned} \tag{4.4.12}$$

where we have used (4.4.11) in the last step. Now we estimate the Fourier coefficients $\widehat{(f^{(\ell)})}_k$

of the ℓth continuous derivative:

$$\begin{aligned}
|(\widehat{f^{(\ell)}})_k| &= \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f^{(\ell)}(x)\, e^{-ikx}\, \mathrm{d}x \right| \\
&\leq \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} |f^{(\ell)}(x)|\, \underbrace{|e^{-ikx}|}_{\leq 1}\, \mathrm{d}x \\
&\leq \left(\sup_{x\in[-\pi,\pi]} |f^{(\ell)}(x)| \right) \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} 1\, \mathrm{d}x \\
&= \sqrt{2\pi} \sup_{x\in[-\pi,\pi]} |f^{(\ell)}(x)| \\
&= \sqrt{2\pi}\, \|f^{(\ell)}\|_{C([-\pi,\pi])}. \qquad (4.4.13)
\end{aligned}$$

Applying (4.4.13) in (4.4.12) and using the error estimate derived from the integral test (for the convergence of a series of real numbers) yields

$$\begin{aligned}
\sup_{x\in[-\pi,\pi]} |f(x) - S_n f(x)| &\leq \frac{1}{\sqrt{2\pi}} \sum_{\substack{k\in\mathbb{Z},\\ |k|>n}} \frac{|(\widehat{f^{(\ell)}})_k|}{|k|^\ell} \\
&\leq \frac{1}{\sqrt{2\pi}} \sum_{\substack{k\in\mathbb{Z},\\ |k|>n}} \frac{\sqrt{2\pi}\, \|f^{(\ell)}\|_{C([-\pi,\pi])}}{|k|^\ell} \\
&\leq \|f^{(\ell)}\|_{C([-\pi,\pi])} \sum_{\substack{k\in\mathbb{Z},\\ |k|>n}} \frac{1}{|k|^\ell} \\
&\leq 2\, \|f^{(\ell)}\|_{C([-\pi,\pi])} \sum_{k=n+1}^{\infty} \frac{1}{k^\ell} \\
&\leq 2\, \|f^{(\ell)}\|_{C([-\pi,\pi])} \int_n^\infty \frac{1}{x^\ell}\, \mathrm{d}x \\
&= 2\, \|f^{(\ell)}\|_{C([-\pi,\pi])} \left[\frac{(-1)}{(\ell-1)\, x^{\ell-1}} \right]_n^\infty \\
&= \frac{2\, \|f^{(\ell)}\|_{C([-\pi,\pi])}}{(\ell-1)} \frac{1}{n^{\ell-1}},
\end{aligned}$$

which proves the uniform error estimate (4.4.10). □

Exercise 56 *Let $f : [-\pi,\pi] \to \mathbb{R}$ be defined by $f(x) = (x^2 - \pi^2)^3$, and extend f with 2π-periodicity to $\mathbb{R}$. Use Corollary 4.12 to derive an explicit error estimate for the uniform approximation of f on $[-\pi,\pi]$ by the partial sums $S_n f$ of its Fourier series. Give a lower bound N on n such that*

$$\|f - S_n f\|_{C([-\pi,\pi])} = \sup_{x\in[-\pi,\pi]} |f(x) - S_n f(x)| \leq 10^{-3} \qquad \textit{for all } n \geq N.$$

4.5 Special Cases and Examples

The next two propositions consider special cases, namely the Fourier series of an even function and the Fourier series of an odd function in $L_2(\mathbb{T})$, respectively. We will prove one of these propositions and leave the proof of the other one as an exercise.

Definition 4.13 (even function and odd function)

(i) A function $f : \mathbb{R} \to \mathbb{R}$ is called ***even*** *(or an* ***even function****) if*

$$f(-x) = f(x) \qquad \text{for all } x \in \mathbb{R}.$$

(ii) A function $f : \mathbb{R} \to \mathbb{R}$ is called ***odd*** *(or an* ***odd function****) if*

$$f(-x) = -f(x) \qquad \text{for all } x \in \mathbb{R}.$$

Example 4.14 (odd and even functions)

(a) $\sin(kx)$, with fixed $k \in \mathbb{Z}$ is an odd function.

(b) $\cos(kx)$, with fixed $k \in \mathbb{Z}$ is an even function.

(c) $f(x) = x^2$ is an even function, and $f(x) = x^3$ is an odd function

(d) Constant functions are even functions.

(e) The only function that is both even and odd is the zero function. □

Exercise 57 *Let $k \in \mathbb{N}_0$. Use the addition theorems for trigonometric functions to verify that $\sin(kx)$ is an odd function and that $\cos(kx)$ is an even function.*

Exercise 58 *Show that any linear combination of even functions is also an even function. Likewise show that any linear combination of odd functions is also an odd function.*

Exercise 59 *Consider the set $\Pi(\mathbb{R})$ of all polynomials on $\mathbb{R}$ with complex coefficients.*

(a) Determine the set $\Pi_{\text{odd}}(\mathbb{R})$ of all polynomials in $\Pi(\mathbb{R})$ that are odd.

(b) Likewise determine the set $\Pi_{\text{even}}(\mathbb{R})$ of all polynomials in $\Pi(\mathbb{R})$ that are even.

(c) Give a proof that you have derived the correct sets in (a) and (b).

(d) Is each of the sets $\Pi_{\text{odd}}(\mathbb{R})$ and $\Pi_{\text{even}}(\mathbb{R})$ a linear space? Give a proof of your answer.

The Fourier series of an even real-valued function in $L_2(\mathbb{T})$ is a so-called Fourier cosine series.

Proposition 4.15 (Fourier cosine series of an even function)
*Suppose $f \in C(\mathbb{T})$ is real-valued and an **even function**, that is, $f(-x) = f(x)$ for all $x \in [-\pi, \pi]$. Then, the Fourier coefficients of f are **real-valued** and can be computed with the formula*

$$\widehat{f}_k = \sqrt{\frac{2}{\pi}} \int_0^\pi f(x)\, \cos(kx)\, \mathrm{d}x, \qquad k \in \mathbb{Z}. \tag{4.5.1}$$

*Furthermore, the Fourier series becomes now a **Fourier cosine series**:*

$$f(x) = \frac{1}{\sqrt{2\pi}}\, \widehat{f}_0 + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \widehat{f}_k\, \cos(kx). \tag{4.5.2}$$

The Fourier series of an odd real-valued function in $L_2(\mathbb{T})$ is a so-called Fourier sine series.

Proposition 4.16 (Fourier sine series of an odd function)
*Suppose $f \in C(\mathbb{T})$ is real-valued and an **odd function**, that is, $f(-x) = -f(x)$ for all $x \in [-\pi, \pi]$. Then, the Fourier coefficients of f are **imaginary** and are given by $\widehat{f}_0 = 0$ and*

$$\widehat{f}_k = -i \sqrt{\frac{2}{\pi}} \int_0^\pi f(x)\, \sin(kx)\, \mathrm{d}x, \qquad k \in \mathbb{Z} \setminus \{0\}.$$

*Furthermore, the Fourier series becomes now a **Fourier sine series**:*

$$f(x) = \sqrt{\frac{2}{\pi}}\, i \sum_{k=1}^{\infty} \widehat{f}_k\, \sin(kx) = \sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} \widetilde{f}_k\, \sin(kx),$$

where we have defined the modified real-valued Fourier coefficients $\widetilde{f}_k$ by

$$\widetilde{f}_k := i\, \widehat{f}_k = \sqrt{\frac{2}{\pi}} \int_0^\pi f(x)\, \sin(kx)\, \mathrm{d}x, \qquad k \in \mathbb{N}.$$

Proof of Proposition 4.15: First we show the formula for the Fourier coefficients:

We observe that for $k = 0$, we have $e^{i0x} = \cos(0x) = 1$ and therefore $e_0(x) = (\sqrt{2\pi})^{-1}$. Thus

$$\begin{aligned}
\widehat{f}_0 &= \int_{-\pi}^{\pi} f(x)\, \overline{e_0(x)}\, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\, \mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\pi}^{0} f(x)\, \mathrm{d}x + \int_0^\pi f(x)\, \mathrm{d}x \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\int_{\pi}^{0} f(-y)\, \mathrm{d}y + \int_0^\pi f(x)\, \mathrm{d}x \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\pi} f(-y)\, \mathrm{d}y + \int_0^\pi f(x)\, \mathrm{d}x \right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \left[\int_0^\pi f(y)\,\mathrm{d}y + \int_0^\pi f(x)\,\mathrm{d}x \right] \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi f(x)\,\mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi f(x)\,\cos(0x)\,\mathrm{d}x,
\end{aligned}
$$

where we have used the substitution $y = -x$ in the first integral and later-on $f(-y) = f(y)$ since f is even.

For $k \in \mathbb{Z} \setminus \{0\}$ we proceed as follows: Splitting the integral into two integrals over $[-\pi, 0]$ and $[0, \pi]$, respectively, substituting subsequently $y = -x$ for in the first integral, and using the fact that f is even, we find

$$
\begin{aligned}
\widehat{f}_k &= \int_{-\pi}^{\pi} f(x)\,\overline{e_k(x)}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\,e^{-ikx}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_{-\pi}^{0} f(x)\,e^{-ikx}\,\mathrm{d}x + \int_0^\pi f(x)\,e^{-ikx}\,\mathrm{d}x \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[-\int_{\pi}^{0} f(-y)\,e^{iky}\,\mathrm{d}y + \int_0^\pi f(x)\,e^{-ikx}\,\mathrm{d}x \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\pi} f(-y)\,e^{iky}\,\mathrm{d}y + \int_0^\pi f(x)\,e^{-ikx}\,\mathrm{d}x \right] \\
&= \frac{1}{\sqrt{2\pi}} \left[\int_0^{\pi} f(y)\,e^{iky}\,\mathrm{d}y + \int_0^\pi f(x)\,e^{-ikx}\,\mathrm{d}x \right] \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\pi f(x)\,\left[e^{ikx} + e^{-ikx} \right]\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\pi f(x)\,2\,\cos(kx)\,\mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi f(x)\,\cos(kx)\,\mathrm{d}x,
\end{aligned}
$$

where we have used Euler's formula $e^{\pm ikx} = \cos(kx) \pm i\,\sin(kx)$ in the second last step.

Thus we have proved the formula (4.5.1) for the Fourier coefficients $\widehat{f}_k$ for all $k \in \mathbb{Z}$.

To prove the formula (4.5.2) for the Fourier series, we use the new formula (4.5.1) for the Fourier coefficients that we have just derived. We observe that since $\cos(kx) = \cos(-kx)$, we clearly have $\widehat{f}_k = \widehat{f}_{-k}$ for all $k \in \mathbb{Z}$. As f is in $L_2(\mathbb{T})$, we have (in the $L_2(\mathbb{T})$-sense)

$$
f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \widehat{f}_k\,e^{ikx}
$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}}\,\widehat{f}_0\,e^{i0x} + \frac{1}{\sqrt{2\pi}}\sum_{k=-\infty}^{-1}\widehat{f}_k\,e^{ikx} + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\widehat{f}_k\,e^{ikx} \\
&= \frac{1}{\sqrt{2\pi}}\,\widehat{f}_0 + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\widehat{f}_{-k}\,e^{-ikx} + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\widehat{f}_k\,e^{ikx} \\
&= \frac{1}{\sqrt{2\pi}}\,\widehat{f}_0 + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\widehat{f}_k\,e^{-ikx} + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\widehat{f}_k\,e^{ikx} \\
&= \frac{1}{\sqrt{2\pi}}\,\widehat{f}_0 + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\widehat{f}_k\,\left[e^{-ikx}+e^{ikx}\right] \\
&= \frac{1}{\sqrt{2\pi}}\,\widehat{f}_0 + \frac{1}{\sqrt{2\pi}}\sum_{k=1}^{\infty}\widehat{f}_k\,\left[2\,\cos(kx)\right] \\
&= \frac{1}{\sqrt{2\pi}}\,\widehat{f}_0 + \sqrt{\frac{2}{\pi}}\sum_{k=1}^{\infty}\widehat{f}_k\,\cos(kx),
\end{aligned}
$$

which proves the formula (4.5.2) for the Fourier series. □

Exercise 60 *Prove Proposition 4.16.*

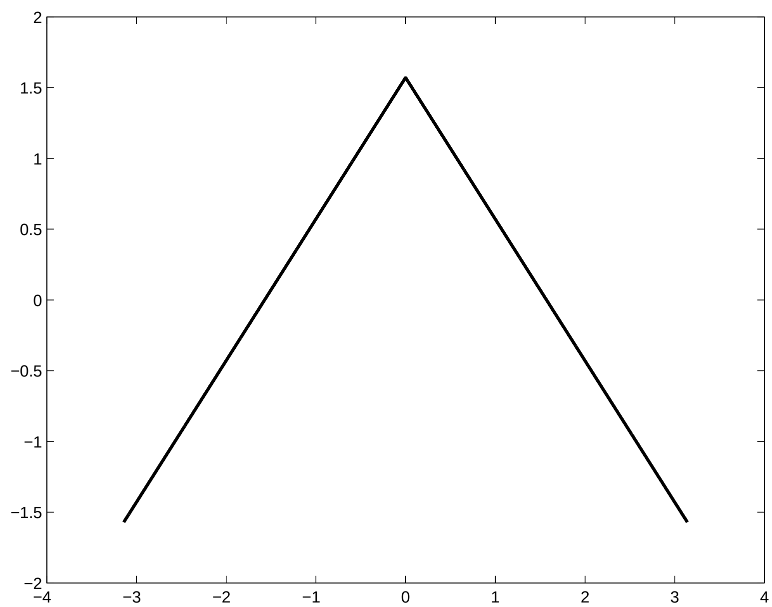

Figure 4.2: The function $h(x) = \frac{\pi}{2} - |x|$.

Example 4.17 (Fourier cosine series)
Let us consider the function h defined on $[-\pi, \pi]$ by

$$h(x) := \frac{\pi}{2} - |x| \qquad \text{for } x \in [-\pi, \pi],$$

and extended periodically to $\mathbb{R}$. This function is in $C(\mathbb{T})$ and $L_2(\mathbb{T})$. The function h is plotted in Figure 4.2.

Since clearly $h(x) = h(-x)$ for all $x \in \mathbb{R}$, the function h is even, and we can use Proposition 4.15 to compute its Fourier coefficients: For $k = 0$, we have (using $\cos(0x) = 1$)

$$\widehat{h}_0 = \sqrt{\frac{2}{\pi}} \int_0^\pi h(x)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \int_0^\pi \left(\frac{\pi}{2} - x\right)\,\mathrm{d}x = \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2}\,x - \frac{1}{2}\,x^2\right]_0^\pi = \sqrt{\frac{2}{\pi}} \left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] = 0.$$

For $k \neq 0$ we have, using $\sin(\ell\pi) = 0$ for all $\ell \in \mathbb{Z}$ and integration by parts,

$$\begin{aligned}
\widehat{h}_k &= \sqrt{\frac{2}{\pi}} \int_0^\pi \left(\frac{\pi}{2} - x\right) \cos(kx)\,\mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \int_0^\pi \frac{\pi}{2}\,\cos(kx)\,\mathrm{d}x - \sqrt{\frac{2}{\pi}} \int_0^\pi x\,\cos(kx)\,\mathrm{d}x \\
&= \sqrt{\frac{2}{\pi}} \left[\frac{\pi}{2}\,\frac{1}{k}\,\sin(kx)\right]_0^\pi - \sqrt{\frac{2}{\pi}} \left(\left[\frac{1}{k}\,x\,\sin(kx)\right]_0^\pi - \frac{1}{k} \int_0^\pi \sin(kx)\,\mathrm{d}x\right) \\
&= 0 - 0 - \sqrt{\frac{2}{\pi}} \left[\frac{1}{k^2}\,\cos(kx)\right]_0^\pi \\
&= -\sqrt{\frac{2}{\pi}}\,\frac{1}{k^2}\,\big[\cos(k\pi) - \cos(0)\big] \\
&= \sqrt{\frac{2}{\pi}}\,\frac{1}{k^2}\,\big[1 - (-1)^k\big] \\
&= \begin{cases} \dfrac{2\sqrt{2}}{\sqrt{\pi}}\,\dfrac{1}{k^2} & \text{if} \quad k \text{ is odd,} \\[2ex] 0 & \text{if} \quad k \text{ is even.} \end{cases}
\end{aligned}$$

Thus the Fourier series of h is the Fourier cosine series

$$\begin{aligned}
h(x) &= \frac{\widehat{h}_0}{\sqrt{2\pi}} + \sqrt{\frac{2}{\pi}} \sum_{k=1}^\infty \widehat{h}_k\,\cos(kx) \\
&= 0 + \sqrt{\frac{2}{\pi}} \sum_{\ell=0}^\infty \frac{2\sqrt{2}}{\sqrt{\pi}\,(2\ell+1)^2}\,\cos\big((2\ell+1)x\big) \\
&= \frac{4}{\pi} \sum_{\ell=0}^\infty \frac{1}{(2\ell+1)^2}\,\cos\big((2\ell+1)x\big),
\end{aligned}$$

where in the second step we have made the substitution $k = 2\ell + 1$, $\ell \in \mathbb{N}_0$, to represent all positive odd integers (for which the Fourier coefficients are different from zero). Figure 4.3 shows some of the partial sums $S_n h$ of the Fourier series of $h(x) = \frac{\pi}{2} - |x|$. □

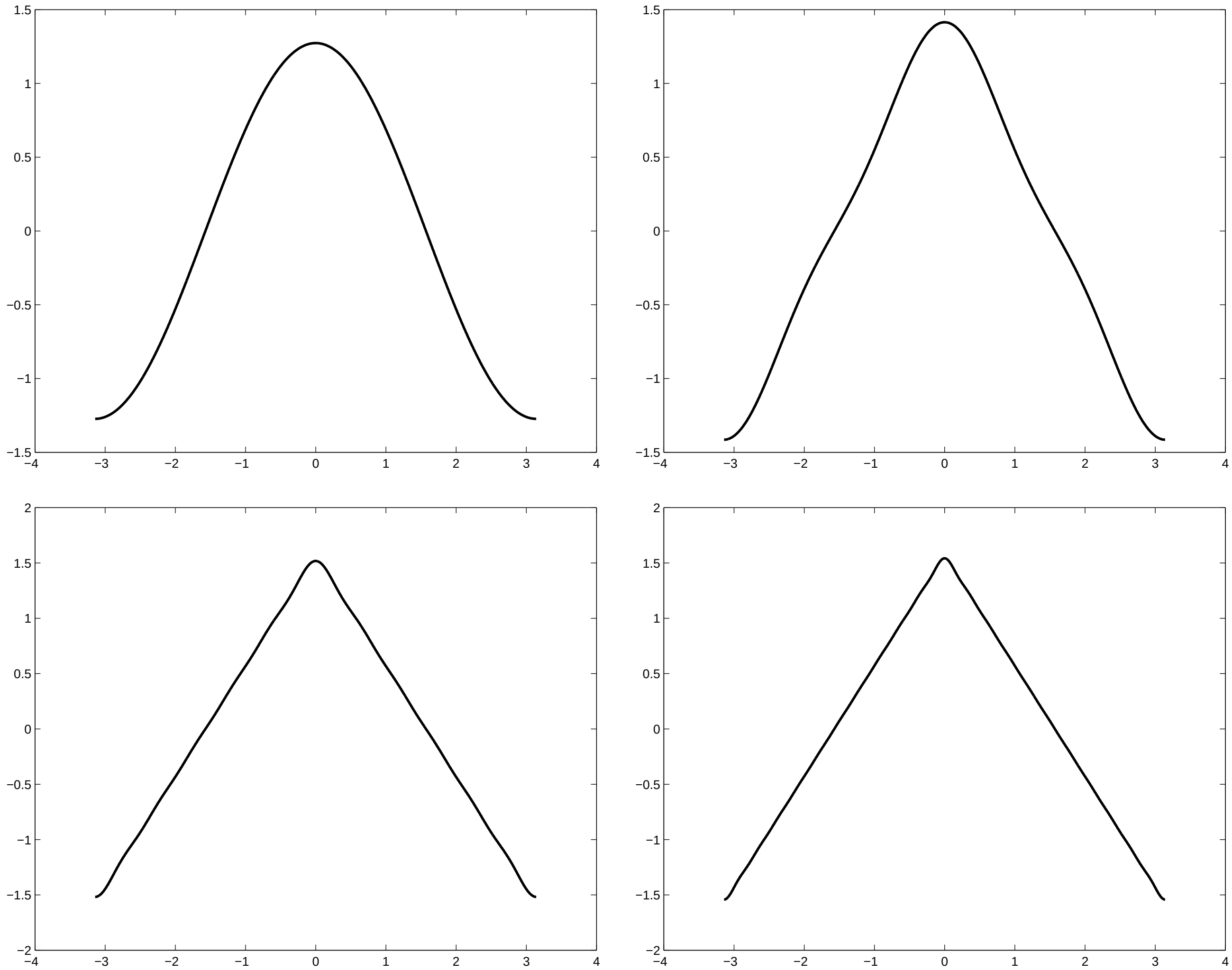

Figure 4.3: Partial sums $S_n h$ for $n = 1$ (top left), $n = 3$ (top right), $n = 11$ (bottom left), and $n = 21$ (bottom right), where $h(x) := \frac{\pi}{2} - |x|$.

Example 4.18 (Gibbs phenomenon)
The **sawtooth function** defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in \{-\pi, \pi\}, \\ x & \text{if} \quad x \in (-\pi, \pi), \end{cases}$$

and extended with 2π-periodicity to $\mathbb{R}$, is an odd discontinuous function with finite jumps at $x = \pi + 2\pi\,\ell$, $\ell \in \mathbb{Z}$. Indeed, we have $f(-\pi) = -f(\pi) = 0 = f(\pi)$ and

$$f(-x) = -x = -f(x) \qquad \text{for all } x \in (-\pi, \pi).$$

Due to the 2π-periodicity, this implies that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. From

$$\|f\|^2_{L_2([-\pi,\pi])} = \int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x = \int_{-\pi}^{\pi} x^2 \,\mathrm{d}x = \left[\frac{1}{3}\,x^3\right]_{-\pi}^{\pi} = \frac{2\,\pi^3}{3} < \infty,$$

the sawtooth function f is clearly in $L_2(\mathbb{T})$. From Proposition 4.16 and Example 3.56, we have that the Fourier coefficients of f are given by $\widehat{f}_0 = 0$ and

$$\widehat{f}_k = \frac{\sqrt{2\pi}\,i}{k}\,(-1)^k, \qquad k \in \mathbb{N},$$

and that the Fourier series is the Fourier sine series given by

$$f(x) = \sum_{k=1}^{\infty} \frac{2\,(-1)^{k+1}}{k}\,\sin(kx),$$

where the equality holds in the $L_2([-\pi,\pi])$ sense.

It is easy to verify that the Fourier series does not converge uniformly on $[-\pi,\pi]$ to the function f. Indeed, the partial sums $S_n f$ are trigonometric polynomials and thus are, in particular, continuous functions on $[-\pi,\pi]$. We know from 'Further Analysis' that the uniform limit of a sequence of continuous functions is also continuous. However, the function f is discontinuous at $x=-\pi$ and $x=\pi$, and therefore the sequence of partial sums (and hence of the Fourier series) cannot converge uniformly on $[-\pi,\pi]$.

The plots in Figure 4.4 shows from left to right the original function and its approximation $S_{50} f$. We observe strong oscillations of $S_{50} f$ towards the points $2\pi\,\ell$, $\ell \in \mathbb{N}$, which are due to the jump discontinuity of the function f. The plots illustrate well that the Fourier series does not converge uniformly on $[-\pi,\pi]$ to the function f. □

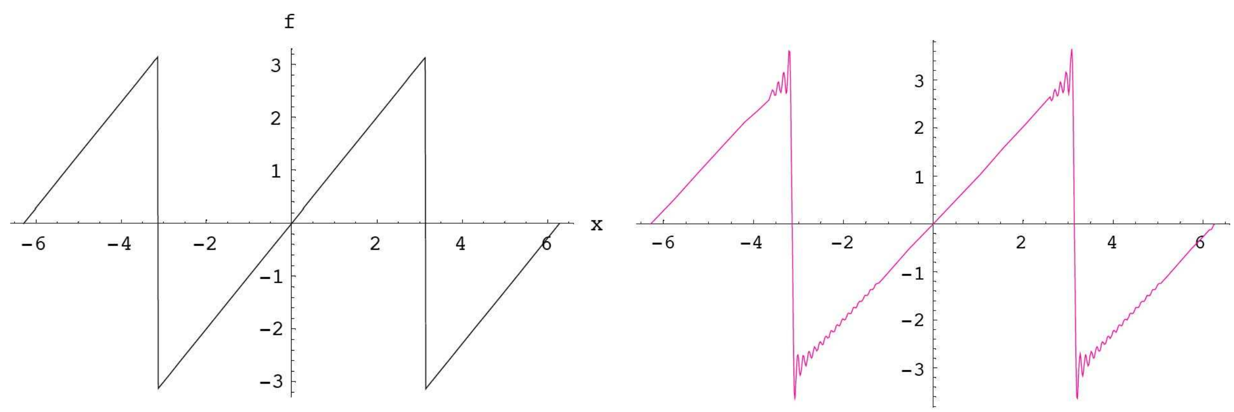

Figure 4.4: The Sawtooth function and its approximations.

Exercise 61 *Consider the following odd 2π-periodic function in $L_2(\mathbb{T})$*

$$f(x) = \begin{cases} 0 & \text{if } x \in \{-\pi,\pi\}, \\ 1 - \dfrac{2}{\pi}(x+\pi) & \text{if } x \in (-\pi,0), \\ 1 - \dfrac{2}{\pi}x & \text{if } x \in [0,\pi). \end{cases}$$

which is extended with 2π-periodicity to $\mathbb{R}$.

(a) Sketch the function f.

(b) Verify that the function f is in $L_2(\mathbb{T})$.

(c) Verify that f is an odd function, that is, verify $f(-x) = f(x)$ for all $x \in [-\pi,\pi]$. (Due to the 2π-periodicity, it is enough to verify the condition for an odd function on the symmetric interval $[-\pi,\pi]$.)

(d) Compute the Fourier coefficients of f.

(e) Compute the Fourier sine series of f.

(f) Explain in which sense the Fourier series converges.

(g) Does the Fourier series converge pointwise to f at all points $x \in [-\pi, \pi]$? Does it converge uniformly on $[\pi, \pi]$ to f?

4.6 The Discrete Fourier Transform

Under the **Fourier transform** we understand, given a function $f \in L_2(\mathbb{T})$, the computation of an approximation of f given in form of a **partial sum of the Fourier series** of f, that is,

$$S_n f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-n}^{n} \widehat{f}_k \, e^{ikx}, \tag{4.6.1}$$

where the **Fourier coefficients** $\widehat{f}_k$ are given by

$$\widehat{f}_k = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) \, e^{ikx} \, \mathrm{d}x, \qquad k \in \mathbb{Z}. \tag{4.6.2}$$

From the material in the previous chapter, we know that $S_n f$ is the $L_2([-\pi, \pi])$**-orthogonal projection of f onto the space**

$$U_n := \operatorname{span} \left\{ e_k(x) := \frac{1}{\sqrt{2\pi}} e^{ikx} \; : \; k = -n, \ldots, n \right\}$$

and that $S_n f$ is also the **best approximation of f in the space** U_n. This means that $S_n f$ is the 'perfect choice' for an approximation of f in the space U_n with respect to the $L_2([-\pi, \pi])$ norm $\| \cdot \|_{L_2([-\pi,\pi])}$.

However, in applications the Fourier transform often encounters the **following problems**:

(i) The function f itself is not known in general, but we only have its function values $f(x_j)$ at certain discrete, sampled points x_j. These sampled points are often equidistant.

(ii) The function f might be known but its Fourier coefficients (4.6.2) cannot be expressed in analytic form (that is, the integrals in (4.6.2) cannot be evaluated).

(iii) The evaluation of the partial Fourier sum $S_n f$ is too expensive, since each evaluation needs linear time (that is, once the Fourier coefficients are known, to evaluate $S_n f(x)$ at a point x we need $O(n)$ operations).

These problems were resolved with the invention of the (discrete) Fast Fourier Transform (FFT). The success of the Fourier transform in digital image processing and signal processing is mainly based on the FFT.

We shall now derive a variant of the **(discrete) Fourier transform** which can be used as the starting point for deriving the Fast Fourier Transform (FFT). The key trick is to discretise the integrals in the Fourier coefficients with a **numerical integration rule** that is exact for trigonometric polynomials of high degree. This gives a variant of the (discrete) Fourier transform. The FFT is a method/algorithm for computing the discrete Fourier transform in a fast and computationally cost effective way that uses only $N \ln N$ operations to compute $S_N f$, where the computation of the Fourier coefficients is included in the $N \ln N$ operations.

Lemma 4.19 (trapezoidal rule)
The ***trapezoidal rule*** *for the numerical integration over* $[0, 2\pi]$ *of continuous functions that are* 2π*-periodic is defined by*

$$Q_N(f) := \frac{2\pi}{N} \sum_{j=0}^{N-1} f\left(\frac{2\pi\, j}{N}\right), \qquad f \in C([0,2\pi]). \tag{4.6.3}$$

The rule Q_N *is exact for all trigonometric polynomials of degree* $\leq N-1$*, that is,*

$$Q_N(f) = \int_0^{2\pi} f(x)\,\mathrm{d}x \quad \textit{for all } f \in U_{N-1} := \operatorname{span}\left\{\frac{1}{\sqrt{2\pi}}\, e^{ikx} \,:\, k = -(N-1), \dots, N-1\right\}. \tag{4.6.4}$$

The proof of this well-known statement follows easily with the help of the geometric series.

Proof of Lemma 4.19: First we observe that, due to the linearity of the rule Q_N, it is enough to verify (4.6.4) for a basis of U_{N-1}. Thus it is enough to verify (4.6.4) for $e_k(x) = (\sqrt{2\pi})^{-1}\, e^{ikx}$, $k = -(N-1), \dots, N-1$. We first compute the integral

$$\int_0^{2\pi} e_k(x)\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{ikx}\,\mathrm{d}x = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^{2\pi} 1\,\mathrm{d}x = \sqrt{2\pi} & \text{if} \quad k = 0, \\ \left[\dfrac{1}{\sqrt{2\pi}}\, \dfrac{e^{ikx}}{ik}\right]_0^{2\pi} = 0 & \text{if} \quad k \neq 0. \end{cases}$$

Next we evaluate the trapezoidal rule, using the geometric series,

$$\begin{aligned} Q_N(e_k) &= \frac{2\pi}{N} \sum_{j=0}^{N-1} e_k\left(\frac{2\pi\, j}{N}\right) = \frac{\sqrt{2\pi}}{N} \sum_{j=0}^{N-1} \exp\left(ik\, \frac{2\pi\, j}{N}\right) = \frac{\sqrt{2\pi}}{N} \sum_{j=0}^{N-1} \left[\exp\left(\frac{i\, 2\pi\, k}{N}\right)\right]^j \\ &= \begin{cases} \dfrac{\sqrt{2\pi}}{N} \displaystyle\sum_{j=0}^{N-1} 1 = \dfrac{\sqrt{2\pi}}{N}\, N = \sqrt{2\pi} \qquad \text{if} \quad k = 0, \\ \dfrac{\sqrt{2\pi}}{N}\, \dfrac{1 - \left[\exp\left(\frac{i\, 2\pi\, k}{N}\right)\right]^N}{1 - \exp\left(\frac{i\, 2\pi\, k}{N}\right)} = \dfrac{\sqrt{2\pi}}{N}\, \dfrac{1 - e^{\frac{i\, 2\pi\, k}{N} N}}{1 - e^{\frac{i\, 2\pi\, k}{N}}} = \dfrac{\sqrt{2\pi}}{N}\, \dfrac{1 - e^{i\, 2\pi\, k}}{1 - e^{\frac{i\, 2\pi\, k}{N}}} = 0 \\ \qquad \text{if} \quad k \in \{-(N-1), \dots, N-1\} \setminus \{0\}. \end{cases} \end{aligned}$$

Thus we see that indeed

$$Q_N(e_k) = \int_0^{2\pi} e_k(x)\,\mathrm{d}x \qquad \text{for all} \quad k = -(N-1), \dots, N-1,$$

and hence (4.6.4) holds true. □

First we observe that, as functions in $L_2(\mathbb{T})$ are 2π-periodic and are in fact defined on $\mathbb{R}$ (via periodicity), we have that

$$\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x = \int_{0}^{2\pi} g(x)\,\mathrm{d}x \qquad \text{for all } g \in L_2(\mathbb{T}). \tag{4.6.5}$$

Hence the **trapezoidal rule (4.6.3) can also be used to numerically integrate functions in $L_2(\mathbb{T})$ over** $[-\pi,\pi]$. Thus we can use the trapezoidal rule to compute the Fourier coefficients of a given function $f \in L_2(\mathbb{T})$.

$$\begin{aligned} \widehat{f}_k &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x)\,e^{-ikx}\,\mathrm{d}x \;=\; \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} f(x)\,e^{-ikx}\,\mathrm{d}x \\ &\approx \frac{1}{\sqrt{2\pi}}\, Q_N(f\,e^{-ik\cdot}) \;=\; \frac{\sqrt{2\pi}}{N} \sum_{j=0}^{N-1} f\left(\frac{2\pi\,j}{N}\right) \exp\left(-ik\,\frac{2\pi\,j}{N}\right). \end{aligned} \tag{4.6.6}$$

Now let $N = 2n$ and let f be a trigonometric polynomial of degree $\le n-1$, that is, assume that $f \in U_{n-1} = \operatorname{span}\{e_k : k = -(n-1),\ldots,n-1\}$. Then $\widehat{f}_k = 0$ for $|k| > n-1$, and

$$f = \sum_{k=-(n-1)}^{n-1} \widehat{f}_k\, \frac{1}{\sqrt{2\pi}}\, e^{ikx}.$$

For $k = -(n-1)\ldots,n-1$ the $\approx$ in (4.6.6) now actually becomes an equality, since $f(x)\,e^{-ikx}$ is a trigonometric polynomial of degree $\le 2(n-1) < 2n-1 = N-1$ and is hence in $U_{N-1} = U_{2n-1} = \operatorname{span}\{e_k : k = -(2n-1),\ldots,2n-1\}$, and functions in this space are integrated exactly by Q_N (see Lemma 4.19). This means that any $f \in U_{n-1}$ is recovered exactly by computing

$$\big(FT_{n-1}(f)\big)(x) := \frac{1}{2\pi} \sum_{k=-(n-1)}^{n-1} Q_{2n}(f\,e^{ik\cdot})\,e^{ikx}. \tag{4.6.7}$$

From (4.6.7) and (4.6.6) with $N = 2n$, we see that the evaluation of the **discrete Fourier transform** $\big(FT_n(f)\big(x)$ at x (which is exact for functions in $U_{n-1} = \operatorname{span}\{e_k : k = -(n-1),\ldots,n-1\}$) costs

$$\sum_{k=-(n-1)}^{n-1} 2n = (2n-1)\,2n \approx 4\,n^2 = O(n^2)$$

elementary operations. (Here an elementary operation consists of one addition and one multiplication.) For large n, this computational cost is very high, and there is a smarter more efficient algorithm to compute the approximation (4.6.7), the so-called **(discrete) Fast Fourier transform** that will however not be discussed in this course. The FFT is based on considering the special case that $f \in U_{2^{m-1}-1}$ and $N = 2^m$ for some $m \in \mathbb{N}$ and exploiting 'symmetries' to obtain a cost effective code for computing (4.6.7).

4.7 The Weierstrass Approximation Theorem

The Weierstrass approximation theorem is an important theorem in analysis that guarantees that any **continuous function** on a bounded closed interval $[a, b]$ can be **approximated uniformly on** $[a, b]$ **by algebraic polynomials**.

Theorem 4.20 (Weierstrass approximation theorem)
For any $f \in C([a, b])$ and any $\epsilon > 0$ there exists an ***algebraic polynomial*** *p such that*

$$\|f - p\|_{C([a,b])} = \sup_{x\in[a,b]} |f(x) - p(x)| < \epsilon. \tag{4.7.1}$$

We note that the Weierstrass approximation theorem implies that, given $f \in C([a, b])$, there exists a **sequence** $(p_n)_{n\in\mathbb{N}}$ **of algebraic polynomials** such that

$$\lim_{n\to\infty} \|f - p_n\|_{C([a,b])} = \lim_{n\to\infty} \left(\sup_{x\in[a,b]} |f(x) - p_n(x)| \right) = 0. \tag{4.7.2}$$

In other words, the **sequence** $(p_n)_{n\in\mathbb{N}}$ **converges uniformly on** $[a, b]$ **to** f. (To see that there exists a sequence $(p_n)_{n\in\mathbb{N}}$ such that (4.7.2) is satisfied, choose, for any $n \in \mathbb{N}$, in (4.7.1) $\epsilon = 1/n$ and write $p_n := p$ for an algebraic polynomial p satisfying (4.7.1) with $\epsilon = 1/n$. Then $(p_n)_{n\in\mathbb{N}}$ gives a sequence of algebraic polynomials satisfying (4.7.2).)

There are various ways of proving Theorem 4.20. In this course, we will exploit Fejér's theorem (see Theorem 4.4) to prove the Weierstrass approximation theorem in an easy way.

Proof of Theorem 4.20: The proof is given in two steps: Initially we consider $f \in C([a, b])$ that is $(b - a)$-periodic, that is, $f(a) = f(b)$, and exploit Fejér's theorem to prove that f can be approximated uniformly on $[a, b]$ by algebraic polynomials. In a second step, we consider arbitrary $f \in C([a, b])$.

Step 1: Suppose first that $f \in C([a, b])$ is $(b - a)$-periodic, that is, $f(a) = f(b)$, and define f on all of $\mathbb{R}$ by extending it periodically with period $(b - a)$. Mapping the interval $[-\pi, \pi]$ onto $[a, b]$ with the affine linear function

$$\phi(x) = \frac{b - a}{2\pi} (x + \pi) + a$$

(which satisfies $\phi(-\pi) = a$ and $\phi(\pi) = b$), we can define a function in $C(\mathbb{T})$ via

$$g(x) := f\big(\phi(x)\big) = f\left(\frac{b - a}{2\pi} (x + \pi) + a\right).$$

Note that from the properties of f, this function is clearly 2π-periodic, since $g(-\pi) = f(\phi(-\pi)) = f(a) = f(b) = f(\phi(\pi)) = g(\pi)$ and since likewise g is defined periodically on the rest of $\mathbb{R}$.

Fix $\epsilon > 0$, and define $(G_m)_{m\in\mathbb{N}}$, via (4.1.9), by

$$G_m(x) = \int_{-\pi}^{\pi} g(y)\, K_m(x - y)\, \mathrm{d}y,$$

where K_m is the Fejér kernel (see (4.1.6)). The function G_m is a linear combination of the complex trigonometric basis polynomials e^{ikx}, $k = -m \ldots, m$, and is therefore a trigonometric polynomial of degree $\leq m$. From Fejér's Theorem 4.4, the sequence $(G_m)_{m\in\mathbb{N}}$ converges uniformly on $[-\pi, \pi]$ to $g \in C(\mathbb{T})$. Thus there exists a $M = M(\epsilon)$ such that

$$\|g - G_m\|_{C([-\pi,\pi])} = \sup_{x\in[-\pi,\pi]} |g(x) - G_m(x)| < \frac{\epsilon}{2} \qquad \text{for all } m \geq M. \tag{4.7.3}$$

The function G_M is a finite linear combination of the complex trigonometric basis polynomials $e_k(x) = (\sqrt{2\pi})^{-1} e^{ikx}$, $k = -M, \ldots, M$, and the Taylor series of each e^{ikx} converges uniformly on $[-\pi, \pi]$ to e^{ikx} (as e^{ikx} is a holomorphic function). Therefore there exists a polynomial $q(x)$, given by replacing the $e_k(x) = (\sqrt{2\pi})^{-1} e^{ikx}$, $k = -M, \ldots, M$, in $G_M(x)$ by their Taylor polynomials of sufficiently high degree, such that

$$\|G_M - q\|_{C([-\pi,\pi])} = \sup_{x\in[-\pi,\pi]} |G_M(x) - q(x)| < \frac{\epsilon}{2}. \tag{4.7.4}$$

Combining (4.7.3) and (4.7.4), we obtain from the triangle inequality

$$\sup_{x\in[-\pi,\pi]} |g(x) - q(x)| = \|g - q\|_{C([-\pi,\pi])} \leq \|g - G_M\|_{C([-\pi,\pi])} + \|G_M - q\|_{C([-\pi,\pi])} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{4.7.5}$$

Changing the variables back, that is, defining

$$p(t) := q\big(\phi^{-1}(t)\big) = q\left(\frac{2\pi}{b-a}\,(t-a) - \pi\right),$$

defines a polynomial, which approximates f uniformly on $[a, b]$ with accuracy $< \epsilon$. Indeed, from the definition of g and p and from (4.7.5), we find

$$\|f - p\|_{C([a,b])} = \sup_{t\in[a,b]} |f(t) - p(t)| = \sup_{x\in[-\pi,\pi]} \big|f\big(\phi(x)\big) - p\big(\phi(x)\big)\big| = \sup_{x\in[-\pi,\pi]} |g(x) - q(x)| < \epsilon.$$

Step 2: If f is not periodic, then we can choose the linear polynomial

$$s(x) = \frac{f(a) - f(b)}{b-a}\,(x-a) + f(b),$$

which satisfies $s(a) = f(b)$ and $s(b) = f(a)$. The function $\widetilde{f} = f + s$, then satisfies $\widetilde{f}(a) = \widetilde{f}(b) = f(a) + f(b)$ and is therefore periodic with period $[a, b]$. From Step 1, we have the existence of a polynomial $\widetilde{p}$, which approximates $\widetilde{f}$ uniformly on $[a, b]$ with at least ϵ accuracy, that is,

$$\|\widetilde{f} - \widetilde{p}\,\|_{C([a,b])} = \|(f + s) - \widetilde{p}\,\|_{C([a,b])} = \|f - (\widetilde{p} - s)\|_{C([a,b])} < \epsilon,$$

and the polynomial $p = \widetilde{p} - s$ is the desired approximation. □

Chapter 5

Orthogonal Wavelets

In this course we only will consider **orthogonal wavelets**, and will discuss the central ideas of wavelet analysis by studying the **Haar wavelet**. The Haar wavelet is the simplest example of an orthogonal wavelet.

The setting: In this chapter we work always in the space $L_2(\mathbb{R})$, equipped with the $L_2(\mathbb{R})$ inner product

$$\langle f, g \rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,\mathrm{d}x$$

and the corresponding induced norm

$$\|f\|_{L_2(\mathbb{R})} := \sqrt{\langle f, f \rangle_{L_2(\mathbb{R})}} = \left(\int_{\mathbb{R}} |f(x)|^2\,\mathrm{d}x \right)^{1/2}.$$

Aim: We are interested in **constructing orthonormal bases** for $L_2(\mathbb{R})$. We stop a moment and consider the functions e^{ikx}, $k \in \mathbb{Z}$, which served us so well when considering Fourier series. Unfortunately, they are not on $L_2(\mathbb{R})$, since

$$\|e^{ikx}\|_{L_2(\mathbb{R})} = \left(\int_{\mathbb{R}} |e^{ikx}|^2\,\mathrm{d}x \right)^{1/2} = \left(\int_{\mathbb{R}} 1\,\mathrm{d}x \right)^{1/2} = \infty.$$

This illustrates a general issue; functions in $L_2(\mathbb{R})$ need to decay fast enough as $|x| \to \infty$ (in order to guarantee that the norm $\|\cdot\|_{L_2(\mathbb{R})}$ is finite).

Families of functions generated by shifting and scaling: Our introduction of wavelets starts with function families, which are **generated from one single function** ϕ by shifting and scaling. More precisely, we will look at functions of the form

$$\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k)$$

with a fixed function $\phi \in L_2(\mathbb{R})$ satisfying $\|\phi\|_{L_2(\mathbb{R})} = 1$. The first index j will always be the **scaling** index (referring to the scaling of the variable with the factor 2^j), while the second index

k will always be used for **shifting** (for shifting the variable by $-k$). The additional factor $2^{j/2}$ is a normalisation factor and guarantees that

$$\|\phi_{j,k}\|^2_{L_2(\mathbb{R})} = \int_{\mathbb{R}} \left|2^{j/2}\,\phi(2^j x - k)\right|^2 \mathrm{d}x = 2^j \int_{\mathbb{R}} |\phi(2^j x - k)|^2 \,\mathrm{d}x = \int_{\mathbb{R}} |\phi(y)|^2 \,\mathrm{d}y = \|\phi\|^2_{L_2(\mathbb{R})} = 1, \tag{5.0.1}$$

where we have used the substitution $y = 2^j x - k,\ \mathrm{d}y/\,\mathrm{d}x = 2^j$.

For a suitable choice of ϕ, it can be achieved that for each j the shifted functions $\{\phi_{j,k} : k \in \mathbb{Z}\}$ form an $L_2(\mathbb{R})$-orthonormal set. The **scaling function** ϕ introduced later-on will have this property. A scaling function corresponds to an **orthogonal wavelet** ψ, and, for the orthogonal wavelet ψ, the set

$$\left\{\psi_{j,k}(x) := 2^{j/2}\,\psi(2^j x - k) \,:\, j,k \in \mathbb{Z}\right\}$$

will form an $L_2(\mathbb{R})$-orthonormal basis for $L_2(\mathbb{R})$.

Application to the approximation of functions: Since the set of scaled and shifted functions $M = \left\{\psi_{j,k}(x) := 2^{j/2}\,\psi(2^j x - k) \,:\, j,k \in \mathbb{Z}\right\}$ forms an $L_2(\mathbb{R})$-orthonormal basis of $L_2(\mathbb{R})$, every function f in $L_2(\mathbb{R})$ has a unique representation of the form

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{j,k}\,\psi_{j,k},$$

where the coefficients are given by $c_{j,k} = \langle f, \psi_{j,k}\rangle_{L_2(\mathbb{R})}$. For the purposes of computing an **approximation** of f, we can ignore coefficients $c_{j,k}$ that are smaller than a certain threshold.

Localisation: We note here that the scaling function and the wavelet (and also their shifted and scaled copies) have to localise, since they are in $L_2(\mathbb{R})$. (By saying that a function f localises, we mean that the 'majority' of the area under the graph of f is concentrated on some finite interval $[a,b]$ and that the area under the graph of f on $\mathbb{R} \setminus [a,b]$ is very small and hence 'almost negligible'. An example of a function with this property is the Gaussian distribution $f(x) = e^{-x^2/2}$.) The localisation plays a crucial role in wavelet analysis. If the area under the graph of ϕ is concentrated on the interval $[a,b]$, then the area under the graph of the shifted copy $\phi(x-k)$ is concentrated on $[a+k, b+k]$, whereas the area under the graph of the scaled copy $\phi(2^j x)$ is concentrated on $[2^{-j}a, 2^{-j}b]$. Thus shifting 'shifts the area of localisation' and scaling 'scales the area of localisation'.

Actually this chapter gives only a glimpse into some ideas of the fascinating topic of wavelets, and we will explore the ideas mentioned above for the Haar wavelet. While this shows the main ideas of wavelets analysis for the simplest example of an orthogonal wavelet, much of the complicated mathematics behind the construction of wavelets remains hidden. A key idea to constructing wavelets is to use the continuous Fourier transform (not discussed in this course)

$$\widehat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)\, e^{-i\omega x}\,\mathrm{d}x, \qquad f \in L_1(\mathbb{R}),$$

which can be extended to a bijection of $L_2(\mathbb{R})$. By alternately working with the functions themselves or their Fourier transforms, wavelets and scaling functions can be constructed by specifying their Fourier transforms with certain properties. However, this goes beyond the scope of this course and would need to be discussed in a course focussing solely on wavelets.

5.1 Introduction to Orthogonal Wavelets

Now we give the formal definition of an **(orthogonal) wavelet**.

Definition 5.1 ((orthogonal) wavelet)
*A function $\psi \in L_2(\mathbb{R})$ is called an **(orthogonal) wavelet** if the family of functions $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$, defined by*

$$\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k), \qquad j,k \in \mathbb{Z}, \tag{5.1.1}$$

*is an $L_2(\mathbb{R})$-**orthonormal basis of** $L_2(\mathbb{R})$.*

Let us consider an example.

Example 5.2 (Haar wavelet)
Let ψ be the **Haar wavelet**, defined by

$$\psi(x) = \phi(2x) - \phi(2x-1), \qquad \text{where} \qquad \phi(x) = \chi_{[0,1)}(x).$$

Here $\chi_{[0,1)}$ is the characteristic function of the interval $[0,1)$, defined by $\chi_{[0,1)}(x) := 1$ if $x \in [0,1)$ and $\chi_{[0,1)}(x) := 0$ if $x \in \mathbb{R} \setminus [0,1)$. More precisely, the Haar wavelet is given by

$$\psi(x) = \begin{cases} 1 & \text{if} \quad x \in \left[0, \frac{1}{2}\right), \\ -1 & \text{if} \quad x \in \left[\frac{1}{2}, 1\right), \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus [0,1). \end{cases}$$

We note that

$$\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = \int_0^{1/2} 1\,\mathrm{d}x + \int_{1/2}^{1} (-1)\,\mathrm{d}x = \frac{1}{2} - \frac{1}{2} = 0.$$

The function $\phi(x) = \chi_{[0,1)}(x)$ is also referred to as the **Haar scaling function**, and we will come back to the Haar scaling function later. □

We note that the Haar wavelet and the Haar scaling function in the previous example are localised in the sense that they have zero values outside the interval $[0,1)$. In mathematical terminology they have compact support $[0,1]$.

Definition 5.3 (support of a function)
*Let $f : \mathbb{R} \to \mathbb{C}$ be a complex-valued function on $\mathbb{R}$. The **support** $\operatorname{supp}(f)$ **of the function** f is defined as the closure (in $\mathbb{R}$ with the absolute value norm $|\cdot|$) of the set of those points where f has non-zero values, that is,*

$$\operatorname{supp}(f) := \overline{\{x \in D \,:\, f(x) \neq 0\}}.$$

*If the closed set $\operatorname{supp}(f)$ is bounded, then it is compact, and we say that f has **compact support**.*

Example 5.4 (support of functions)

(a) The Haar scaling function $\phi(x) = \chi_{[0,1)}(x)$ and the Haar wavelet $\psi(x) = \phi(2x) - \phi(2x-1)$ have both the compact support

$$\mathrm{supp}\,(\phi) = \mathrm{supp}\,(\psi) = [0,1].$$

(b) The function $f(x) = \sin x$ has the support $\mathrm{supp}\,(f) = \mathbb{R}$. This function does not have compact support. □

Exercise 62 *Find the support of the following functions. Which of these functions do have compact support? Explain your results.*

(a) $f(x) = \cos(x)$, (b) $g(x) = \chi_{(-10,-7]}(x) - \chi_{[1,2)}(x) + \chi_{(3,4)}(x)$, (c) $h(x) = \begin{cases} 0 & \text{if } x \le 0, \\ x^3 & \text{if } x > 0. \end{cases}$

Now we come back to the Haar wavelet.

Example 5.5 (Haar wavelet is an (orthogonal) wavelet)
Let ψ be the **Haar wavelet**, defined by $\psi(x) = \phi(2x) - \phi(2\,x-1)$, where $\phi(x) = \chi_{[0,1)}(x)$. Then $\big\{\psi_{j,k}(x) := 2^{j/2}\,\psi(2^j x - k)\,:\, j,k \in \mathbb{Z}\big\}$ forms an $L_2(\mathbb{R})$-orthonormal basis for $L_2(\mathbb{R})$.

For the moment we will only prove that $\big\{\psi_{j,k}(x) := 2^{j/2}\,\psi(2^j x - k)\,:\, j,k \in \mathbb{Z}\big\}$ forms an $L_2(\mathbb{R})$-orthonormal set. The completeness of this $L_2(\mathbb{R})$-orthonormal set (that is, the fact that this $L_2(\mathbb{R})$-orthonormal set is an orthonormal basis for $L_2(\mathbb{R})$) will be shown later-on.

Proof that $\{\psi_{j,k}(x) := 2^{j/2}\,\psi(2^j x - k)\,:\, j,k \in \mathbb{Z}\}$ *forms an* $L_2(\mathbb{R})$*-orthonormal set in* $L_2(\mathbb{R})$*:* Since $|\psi(x)| = 1$ for all $x \in [0,1)$ and $|\psi(x)| = 0$ elsewhere, we clearly have

$$\|\psi\|_{L_2(\mathbb{R})} = \left(\int_{\mathbb{R}} |\psi(x)|^2\,\mathrm{d}x\right)^{1/2} = \left(\int_0^1 1\,\mathrm{d}x\right)^{1/2} = \sqrt{[x]_0^1} = 1.$$

From the general properties of the scaled and shifted copies $\psi_{j,k}$ (see (5.0.1)) we have

$$\|\psi_{j,k}\|_{L_2(\mathbb{R})} = 1 \qquad \text{for all } j \in \mathbb{Z} \text{ and all } k \in \mathbb{Z},$$

so that the family $\{\psi_{j,k}(x) := 2^{j/2}\,\psi(2^j x - k)\,:\, j,k \in \mathbb{Z}\}$ is normalised. Orthogonality can be proved as follows. We have to show that

$$\langle \psi_{j,k}, \psi_{j',k'} \rangle_{L_2(\mathbb{R})} = \delta_{j,j'}\,\delta_{k,k'} = \begin{cases} 1 & \text{if } j = j' \text{ and } k = k', \\ 0 & \text{else.} \end{cases}$$

First of all, note that (since $\mathrm{supp}\,(\phi) = \mathrm{supp}\,(\chi_{[0,1)}) = [0,1]$) the support of $\psi_{j,k}$ is given by

$$\mathrm{supp}\,(\psi_{j,k}) = \big[2^{-j}k, 2^{-j}(k+1)\big].$$

Hence, if $j = j'$ but $k \ne k'$ then the functions $\psi_{j,k}$ and $\psi_{j',k'}$ have essentially (that is, apart from boundary points) disjoint support. Thus if $j = j'$ but $k \ne k'$ then $\psi_{j,k}(x)\,\overline{\psi_{j',k'}(x)} = 0$ for all $x \in \mathbb{R}$, and hence

$$\langle \psi_{j,k}, \psi_{j',k'} \rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} \psi_{j,k}(x)\,\overline{\psi_{j',k'}(x)}\,\mathrm{d}x = \int_{\mathbb{R}} 0\,\mathrm{d}x = 0.$$

Finally, if $j \neq j'$, we may assume without loss of generality that for $j' < j$. Then there are two possibilities: Either $\psi_{j,k}$ and $\psi_{j',k'}$ have again essentially disjoint support (apart from possibly boundary points) in which case the inner product is zero. Or alternatively the support of $\psi_{j,k}$ is contained in an interval where $\psi_{j',k'}$ does not change sign, that is, $\psi_{j,k}(x)\,\overline{\psi_{j',k'}(x)} = 2^{j'/2}\,\psi_{j,k}(x)$ for all $x \in \mathbb{R}$ or $\psi_{j,k}(x)\,\overline{\psi_{j',k'}(x)} = -2^{j'/2}\,\psi_{j,k}(x)$ for all $x \in \mathbb{R}$. We have

$$\int_{\mathbb{R}} \psi_{j,k}(x)\,\mathrm{d}x = 2^{j/2} \int_{\mathbb{R}} \psi(2^j x - k)\,\mathrm{d}x = 2^{-j/2} \int_{\mathbb{R}} \psi(y)\,\mathrm{d}y = 2^{-j/2} \left(\int_0^{1/2} 1\,\mathrm{d}y - \int_{1/2}^{1} 1\,\mathrm{d}y \right) = 0,$$

where we have used the substitution $y = 2^j x - k$, $\mathrm{d}y/\,\mathrm{d}x = 2^j$. This implies that also $\langle \psi_{j,k}, \psi_{j',k'} \rangle_{L_2(\mathbb{R})} = 0$ if $j \neq j'$ and if $\psi_{j,k}$ and $\psi_{j',k'}$ do not have disjoint support. □

If we represent a function $f \in L_2(\mathbb{R})$ as a series with respect to the $L_2(\mathbb{R})$-orthonormal basis $\{\psi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k) \,:\, j,k \in \mathbb{Z}\}$ generated by the Haar wavelet ψ, then we have

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle_{L_2(\mathbb{R})}\, \psi_{j,k}(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle_{L_2(\mathbb{R})}\, 2^{j/2}\, \psi(2^j x - k). \tag{5.1.2}$$

If we fix the index j and only consider the inner sum, then we have an infinite sum of shifted copies of the function

$$\psi_{j,0}(x) = 2^{j/2}\,\psi(2^j x) = \begin{cases} 2^{j/2} & \text{if} \quad x \in \left[0, 2^{-(j+1)}\right), \\ -2^{j/2} & \text{if} \quad x \in \left[2^{-(j+1)}, 2^{-j}\right), \\ 0 & \text{if} \quad x \in \mathbb{R} \setminus [0, 2^{-j}). \end{cases}$$

Clearly $\mathrm{supp}\,(\psi_{j,0}) = [0, 2^{-j}]$. We see that the compact support of

$$\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k) = 2^{j/2}\,\psi\big(2^j (x - 2^{-j} k)\big)$$

(which is just the function $\psi_{j,0}$ shifted by $2^{-j}k$) is just $\mathrm{supp}\,(\psi_{j,k}) = [2^{-j}k, 2^{-j}\,(k+1)]$. However, if we change j, then the support of the $\psi_{j,k}$ becomes narrower as j increases and becomes wider as j decreases. Thus intuitively, the contributions in (5.1.2) for small j are used to model **long wavelength (low frequency)** features of the signal f and the contributions in (5.1.2) for large j are used to model **short wavelength (high frequency)** features of the signal f.

5.2 Multiresolution Analysis for the Haar Wavelet

In this section we will introduce the concept of a **multiresolution analysis**. First we will encounter all definitions and results for the special case of the Haar wavelet and the Haar scaling function. In Section 5.3 we will then define the concept of a multiresolution analysis as an abstract concept and derive some conclusions.

Starting with the **Haar scaling function** $\phi(x) = \chi_{[0,1)}(x)$ and its shifted and scaled versions

$$\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k), \qquad j,k \in \mathbb{Z}, \tag{5.2.1}$$

we define the following closed subspaces of $L_2(\mathbb{R})$.

Definition 5.6 (scale spaces V_j for the Haar scaling function)
Let $\phi(x) = \chi_{[0,1)}(x)$ be the Haar scaling function. Let $\{\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k) \,:\, j,k \in \mathbb{Z}\}$ denote the family of its scaled and shifted versions. For each $j \in \mathbb{Z}$, the ***scale space*** *V_j is defined by*

$$V_j := \overline{\operatorname{span}\left\{\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k) \,:\, k \in \mathbb{Z}\right\}}^{\|\cdot\|_{L_2(\mathbb{R})}},$$

where the closure is taken with respect to the $L_2(\mathbb{R})$ norm $\|\cdot\|_{L_2(\mathbb{R})}$.

We note that the elements of V_j consist of step functions which are piecewise constant on the intervals $[2^{-j}k, 2^{-j}(k+1)]$.

The term **scale space** refers to the fact that the functions in V_j are in the closure of the span of shifted copies of the scaled version $\phi_{j,0}(x) = 2^{j/2}\,\phi(2^j x)$ of the scaling function ϕ.

Since $\{\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k) \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal set, by the construction of the scale space V_j, the set $\{\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k) \,:\, k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$**-orthonormal basis for** V_j. We state this as a lemma, since it will play a crucial role in our discussion of the Haar wavelet and the Haar scaling function.

Lemma 5.7 ($\{\phi_{j,k}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_j)
*Let $\phi(x) = \chi_{[0,1)}(x)$ be the Haar scaling function. The set $\{\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k) \,:\, k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$-**orthonormal basis** for V_j. Thus for every $f \in V_j$ there exists a sequence of coefficients $\left(c_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$ in $\ell_2(\mathbb{Z})$, given by*

$$c_k^{(j)}(f) := \langle f, \phi_{j,k}\rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\,\overline{\phi_{j,k}(x)}\,\mathrm{d}x, \tag{5.2.2}$$

such that

$$f = \sum_{k\in\mathbb{Z}} c_k^{(j)}(f)\,\phi_{j,k}, \tag{5.2.3}$$

where the series in (5.2.3) converges with respect to $\|\cdot\|_{L_2(\mathbb{R})}$ and where the equality in (5.2.3) holds in the $L_2(\mathbb{R})$ sense.

Here the **linear sequence space** $\ell_2(\mathbb{Z})$ is defined in analogy to $\ell_2(\mathbb{N})$: The linear space $\ell_2(\mathbb{Z})$ is the set of all sequences $c = (c_k)_{k\in\mathbb{Z}}$ in $\mathbb{C}$ for which

$$\|c\|_2 = \|(c_k)_{k\in\mathbb{Z}}\|_2 := \left(\sum_{k\in\mathbb{Z}} |c_k|^2\right)^{1/2} \tag{5.2.4}$$

is finite. The linear space $\ell_2(\mathbb{Z})$ is a Hilbert space with the inner product

$$\langle c, d\rangle_2 = \sum_{k\in\mathbb{Z}} c_k\,\overline{d_k}, \qquad c = (c_k)_{k\in\mathbb{Z}},\ d = (d_k)_{k\in\mathbb{Z}},$$

which also induces the norm (5.2.4).

We note that the formula (5.2.2) for the coefficients, the expansion (5.2.3) and the fact that $\big(c_k^{(j)}(f)\big)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ directly follow from our general knowledge about Hilbert spaces and orthonormal bases (see Theorem 3.62 in Chapter 3).

The next theorem investigates the properties of the scale spaces of the Haar scaling function.

Theorem 5.8 (properties of the scale spaces for the Haar scaling function) *Let $\phi(x) = \chi_{[0,1)}(x)$ be the Haar scaling function, and let V_j, $j \in \mathbb{Z}$, be the scale spaces of the Haar scaling function as in introduced in Definition 5.6. These scale spaces V_j are closed subspaces of $L_2(\mathbb{R})$ with the following properties:*

(i) $V_j \subset V_{j+1}$ *for any* $j \in \mathbb{Z}$.

(ii) For any $j \in \mathbb{Z}$, *we have* $f \in V_j$ *if and only if* $f(2\cdot) \in V_{j+1}$.

(iii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j}^{\|\cdot\|_{L_2(\mathbb{R})}} = L_2(\mathbb{R})$.

(iv) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(v) $\{\phi(x-k) : k \in \mathbb{Z}\}$ *is an* $L_2(\mathbb{R})$*-orthonormal basis for* V_0.

As the properties (i) to (v) are satisfied for the family $\{V_j\}_{j\in\mathbb{Z}}$ of scale spaces V_j of the Haar scaling function, we say that the family $\{V_j\}_{j\in\mathbb{Z}}$ forms a ***multiresolution analysis****.*

We will get a general definition of the term 'multiresolution analysis' in the next section. Essentially it is a collection $\{V_j\}_{j\in\mathbb{Z}}$ of subspaces of $L_2(\mathbb{R})$ such that properties (i) to (v) in Theorem 5.8 are satisfied, where (v) is modified to say that there exists a function $\phi \in L_2(\mathbb{R})$ such that (v) holds.

Proof of Theorem 5.8: We verify the five properties:

(i) Since

$$\chi_{[0,1)}(x) = \chi_{[0,1/2)}(x) + \chi_{[1/2,1)}(x) = \chi_{[0,1)}(2x) + \chi_{[0,1)}(2x-1), \tag{5.2.5}$$

we have

$$\begin{aligned}
\phi_{j,k}(x) &= 2^{j/2}\,\chi_{[0,1)}(2^j x - k) \\
&= 2^{j/2}\Big[\chi_{[0,1)}\big(2\,(2^j x - k)\big) + \chi_{[0,1)}\big(2\,(2^j x - k) - 1\big)\Big] \\
&= \frac{2^{(j+1)/2}}{2^{1/2}}\Big[\chi_{[0,1)}\big(2^{j+1}x - 2k\big) + \chi_{[0,1)}\big(2^{j+1}x - (2k+1)\big)\Big] \\
&= \frac{1}{\sqrt{2}}\Big[\phi_{j+1,2k}(x) - \phi_{j+1,2k+1}(x)\Big],
\end{aligned}$$

which proves that for every $k \in \mathbb{Z}$, the function $\phi_{j,k}$ is in span$\{\phi_{j+1,k} : k \in \mathbb{Z}\}$. Hence any function f in span$\{\phi_{j,k} : k \in \mathbb{Z}\}$ is also in span$\{\phi_{j+1,k} : k \in \mathbb{Z}\}$. From

$$\text{span}\,\{\phi_{j,k} : k \in \mathbb{Z}\} \subset \text{span}\,\{\phi_{j+1,k} : k \in \mathbb{Z}\}$$

we can conclude immediately

$$V_j = \overline{\text{span}\,\{\phi_{j,k} : k \in \mathbb{Z}\}}^{\|\cdot\|_{L_2(\mathbb{R})}} \subset \overline{\text{span}\,\{\phi_{j+1,k} : k \in \mathbb{Z}\}}^{\|\cdot\|_{L_2(\mathbb{R})}} = V_{j+1}.$$

(ii) $\Rightarrow$: Assume that $f \in V_j$; then from Lemma 5.7 there exists a sequence of coefficients $\big(c_k^{(j)}(f)\big)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} c_k^{(j)}(f)\, \phi_{j,k}(x).$$

Hence we have

$$\begin{aligned} g(x) := f(2x) &= \sum_{k\in\mathbb{Z}} c_k^{(j)}(f)\, \phi_{j,k}(2x) \\ &= \sum_{k\in\mathbb{Z}} c_k^{(j)}(f)\, 2^{j/2}\, \phi(2^{j+1}x - k) \\ &= \sum_{k\in\mathbb{Z}} \frac{c_k^{(j)}(f)}{\sqrt{2}}\, \phi_{j+1,k}(x). \end{aligned}$$

Since $\big(c_k^{(j)}(f)\big)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, the sequence $\big((\sqrt{2})^{-1}\, c_k^{(j)}(f)\big)_{k\in\mathbb{Z}}$ is also in $\ell_2(\mathbb{Z})$, and we know from the Riesz-Fischer theorem (see Theorem 3.63) that $g(x) = f(2x)$ is in V_{j+1}.

$\Leftarrow$: Assume that $g(x) = f(2x)$ is in V_{j+1}. Then there exists a sequence of coefficients $\big(c_k^{(j+1)}(g)\big)_{k\in\mathbb{Z}}$ in $\ell_2(\mathbb{Z})$ such that

$$\begin{aligned} f(2x) = g(x) &= \sum_{k\in\mathbb{Z}} c_k^{(j+1)}(g)\, \phi_{j+1,k}(x) \\ &= \sum_{k\in\mathbb{Z}} c_k^{(j+1)}(g)\, 2^{(j+1)/2}\, \phi(2^{j+1}x - k) \\ &= \sum_{k\in\mathbb{Z}} \sqrt{2}\, c_k^{(j+1)}(g)\, \phi_{j,k}(2x). \end{aligned}$$

Since $\big(c_k^{(j+1)}(g)\big)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$, the sequence $\big(\sqrt{2}\, c_k^{(j+1)}(g)\big)_{k\in\mathbb{Z}}$ is also in $\ell_2(\mathbb{Z})$, and the Riesz-Fischer theorem (see Theorem 3.63) tells us that

$$f(x) = \sum_{k\in\mathbb{Z}} \sqrt{2}\, c_k^{(j+1)}(g)\, \phi_{j,k}(x)$$

is in V_j.

(iii) The third property follows from the fact that every function in $L_2(\mathbb{R})$ can be approximated arbitrarily well by step functions (that is, piecewise constant functions). In other words the step functions are dense in $L_2(\mathbb{R})$ with respect to the $L_2(\mathbb{R})$ norm. This is a non-trivial result that is derived during the introduction of the Lebesgue integral. – As the supports of the $\phi_{j,k}$ get arbitrary small as $j \to \infty$ and cover for each fixed j all of $\mathbb{R}$, we can use span $\{\phi_{j,k} \,:\, j,k \in \mathbb{Z}\}$ to approximate any characteristic function $\chi_{[a,b)}$ arbitrarily well, that is, span $\{\phi_{j,k} \,:\, j,k \in \mathbb{Z}\}$ is dense in the set of all step functions with respect to the norm $\|\cdot\|_{L_2(\mathbb{R})}$. Since the set of all step functions is dense in $L_2(\mathbb{R})$, we see that span $\{\phi_{j,k} \,:\, j,k \in \mathbb{Z}\}$ is dense in $L_2(\mathbb{R})$ with respect to the norm $\|\cdot\|_{L_2(\mathbb{R})}$. This together with the imbedding $V_j \subset V_{j+1}$ implies that (iii) holds true.

(iv) For the fourth property note that a function f that belongs to V_j is constant one each of the intervals $[0, 2^{-j})$ and $[-2^{-j}, 0)$. Letting $j \to -\infty$ shows that $f \in \bigcap_{j\in\mathbb{Z}} V_j$ satisfies $f(x) = c_1$ for $x < 0$ and $f(x) = c_2$ for $x \geq 0$ with some constants $c_1, c_2 \in \mathbb{C}$. As $f \in L_2(\mathbb{R})$, that is,

$$\|f\|_{L_2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^2 \,\mathrm{d}x \right)^{1/2} = \left(\int_{-\infty}^{0} |c_1|^2 \,\mathrm{d}x + \int_{0}^{\infty} |c_2|^2 \,\mathrm{d}x \right)^{1/2} < \infty,$$

we have to have $c_1 = c_2 = 0$, that is, f is the zero function.

(v) The fifth property is a special case of Lemma 5.7. for $j = 0$.

This verifies the theorem. □

Later on, we will define a **multiresolution analysis** as a set of function spaces, which satisfy the properties in Theorem 5.8. But for now, we want to take a look at what we can conclude from some of the properties in Theorem 5.8.

First of all, we have $V_0 \subset V_1$, which means in particular that ϕ can be expressed as a linear combination of functions $\phi_{1,k}(x) = \sqrt{2}\,\phi(2x - k)$, $k \in \mathbb{Z}$. For the Haar scaling function, it is easy to see that we have (see also (5.2.5))

$$\phi(x) = \phi(2x) + \phi(2x-1) = \frac{1}{\sqrt{2}} \Big(2^{1/2}\,\phi(2x) - 2^{1/2}\,\phi(2x-1) \Big) = \frac{1}{\sqrt{2}} \Big(\phi_{1,0}(x) - \phi_{1,1}(x) \Big).$$

Such an equation is called a **refinement equation**.

Lemma 5.9 (refinement equation of the Haar scaling function)
*The Haar scaling function $\phi(x) = \chi_{[0,1)}(x)$ satisfies the **refinement equation***

$$\phi(x) = \phi(2x) + \phi(2x-1) = \frac{1}{\sqrt{2}} \Big(\phi_{1,0}(x) - \phi_{1,1}(x) \Big). \tag{5.2.6}$$

From (5.2.6) it follows that

$$\begin{aligned} \phi_{j,k}(x) &= 2^{j/2}\,\phi(2^j x - k) \\ &= 2^{j/2} \Big[\phi(2^{j+1}x - 2k) + \phi(2^{j+1}x - 2k - 1) \Big] \\ &= \frac{1}{\sqrt{2}} \Big[\phi_{j+1,2k}(x) + \phi_{j+1,2k+1}(x) \Big]. \end{aligned}$$

From $V_0 \subset V_1$ (see Theorem 5.8 (i)) and the fact that both spaces are closed subspaces of $L_2(\mathbb{R})$, it follows that V_1 can be decomposed into V_0 and its orthogonal complement in V_1, usually denoted by W_0, such that we have the orthogonal sum $V_1 = V_0 \oplus W_0$. This, of course holds in the more general situation of $V_j \subset V_{j+1}$: we define the **detail space** W_j as the orthogonal complement of V_j in V_{j+1}, that is, we have the orthogonal sum

$$V_{j+1} = V_j \oplus W_j.$$

Definition 5.10 (detail spaces of the Haar scaling function)
Let $\phi(x) = \chi_{[0,1)}(x)$ be the Haar scaling function, and let V_j, $j \in \mathbb{Z}$, be the scale spaces of the Haar scaling function, defined in Definition 5.6. The ***detail space*** *W_j, $j \in \mathbb{Z}$, of the Haar scaling function is defined as the $L_2(\mathbb{R})$*-***orthogonal complement of*** *V_j* ***in*** *V_{j+1}, such that, we have the* ***orthogonal sum***

$$V_{j+1} = V_j \oplus W_j, \qquad j \in \mathbb{Z}.$$

Interestingly, like the scale spaces V_j, the detail spaces W_j are generated by the shifts and scales of only one function, namely the Haar wavelet $\psi(x) = \phi(2x) - \phi(2x-1)$.

Lemma 5.11 ($L_2(\mathbb{R})$-orthonormal bases for V_j and W_j)
Let $\phi(x) = \chi_{[0,1)}(x)$ be the Haar scaling function and $\psi(x) = \phi(2x) - \phi(2x-1)$ be the Haar wavelet. Then the following holds true:

(i) The set $\{\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k) \,:\, k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$-***orthonormal basis*** *for the detail space W_j.*

(ii) The set $\{\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k),\ \phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k) \,:\, k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$-***orthonormal basis*** *for the scale space V_{j+1}.*

Proof of Lemma 5.11: Since the spaces V_j are defined by scaling, it actually suffices to prove this result only for $j = 0$.

By the definition $\psi(x) = \phi(2x) - \phi(2x-1)$ of the Haar wavelet ψ, the shifts $\psi_{0,k}$ of ψ have the representation

$$\psi_{0,k}(x) = \psi(x-k) = \phi\big(2(x-k)\big) - \phi\big(2(x-k)-1\big) = \frac{1}{\sqrt{2}}\Big[\phi_{1,2k}(x) - \phi_{1,2k+1}(x)\Big],$$

and hence the $\psi_{0,k}$ belong to V_1 but not to V_0. Furthermore, they satisfy

$$\langle \phi_{0,k}, \psi_{0,m}\rangle_{L_2(\mathbb{R})} = \big\langle \phi(\cdot - k), \psi(\cdot - m)\big\rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} \phi(x-k)\,\overline{\psi(x-m)}\,\mathrm{d}x = \delta_{k,m},$$

because we have $\phi(x-k)\,\overline{\psi(x-m)} = 0$ for all $x \in \mathbb{R}$ if $k \neq m$ and $\phi(x-k)\,\overline{\psi(x-m)} = \psi(x-k)$ for all $x \in \mathbb{R}$ if $k = m$ and

$$\int_{\mathbb{R}} \psi(x-k)\,\mathrm{d}x = \int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 0.$$

Thus we see that the $\psi_{0,k}$, $k \in \mathbb{Z}$ are orthogonal to V_0, and hence they belong to W_0. That the $\psi_{0,k}$, $k \in \mathbb{Z}$, form an $L_2(\mathbb{R})$-orthonormal set was verified in Example 5.5. Thus we have verified so far that the $\{\psi_{0,k} \,:\, k \in \mathbb{Z}\}$ form an $L_2(\mathbb{R})$-orthonormal set in W_0. To verify (i) it remains to show that this $L_2(\mathbb{R})$-orthonormal set is also an $L_2(\mathbb{R})$-orthonormal basis for W_0, that is, that every $f \in W_0$ has a (unique) representation

$$f = \sum_{k \in \mathbb{Z}} \langle f, \psi_{0,k}\rangle_{L_2(\mathbb{R})}\,\psi_{0,k}.$$

From the previous considerations and the fact that $\{\phi_{0,k} \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_0, we also see that the set $\{\phi_{0,k},\, \psi_{0,k} \,:\, k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$-orthonormal set in V_1. To verify (ii) it remains to show that this $L_2(\mathbb{R})$-orthonormal set is an $L_2(\mathbb{R})$-orthonormal basis for V_1, that is, that every f in V_1 has a (unique) representation

$$f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{0,k}\rangle_{L_2(\mathbb{R})}\, \phi_{0,k} + \sum_{k\in\mathbb{Z}} \langle f, \psi_{0,k}\rangle_{L_2(\mathbb{R})}\, \psi_{0,k}.$$

To complete the proofs of (i) and (ii) we make use of the identities

$$\phi(x) + \psi(x) = 2\,\phi(2x), \tag{5.2.7}$$

$$\phi(x) - \psi(x) = 2\,\phi(2x-1), \tag{5.2.8}$$

whose easy proof is left as an exercise. By replacing in (5.2.7) and (5.2.8) x by $x-k$, we conclude that

$$\phi_{1,2k} = \frac{1}{\sqrt{2}}\big(\phi_{0,k} + \psi_{0,k}\big), \tag{5.2.9}$$

$$\phi_{1,2k+1} = \frac{1}{\sqrt{2}}\big(\phi_{0,k} - \psi_{0,k}\big). \tag{5.2.10}$$

Any $f \in V_1$ has an expansion

$$f = \sum_{k\in\mathbb{Z}} \underbrace{\langle f, \phi_{1,k}\rangle_{L_2(\mathbb{R})}}_{=:\, c_k^{(1)}(f)}\, \phi_{1,k} = \sum_{k\in\mathbb{Z}} c_k^{(1)}(f)\, \phi_{1,k},$$

and substituting (5.2.9) and (5.2.10) for $\phi_{1,k}$ with k even and k odd, respectively, gives

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}} c_{2k}^{(1)}(f)\, \phi_{1,2k} + \sum_{k\in\mathbb{Z}} c_{2k+1}^{(1)}(f)\, \phi_{1,2k+1} \\ &= \sum_{k\in\mathbb{Z}} \frac{c_{2k}^{(1)}(f)}{\sqrt{2}}\, \big(\phi_{0,k} + \psi_{0,k}\big) + \sum_{k\in\mathbb{Z}} \frac{c_{2k+1}^{(1)}(f)}{\sqrt{2}}\, \big(\phi_{0,k} - \psi_{0,k}\big) \\ &= \sum_{k\in\mathbb{Z}} \frac{c_{2k}^{(1)}(f) + c_{2k+1}^{(1)}(f)}{\sqrt{2}}\, \phi_{0,k} + \sum_{k\in\mathbb{Z}} \frac{c_{2k}^{(1)}(f) - c_{2k+1}^{(1)}(f)}{\sqrt{2}}\, \psi_{0,k}. \end{aligned} \tag{5.2.11}$$

This shows that the set $\{\psi_{0,k},\ \phi_{0,k} \,:\, k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$-orthonormal basis for V_1. This proves (ii).

Finally, by definition of W_0, any $f \in W_0$ belongs to V_1 and has therefore a representation (5.2.11). By the definition of W_0 as the $L_2(\mathbb{R})$-orthogonal complement of V_0 in V_1, we have for any $f \in W_0$ that $\langle f, \phi_{0,m}\rangle_{L_2(\mathbb{R})} = 0$ for all $m \in \mathbb{Z}$. Taking in (5.2.11) the inner product with $\phi_{0,m}$ shows that the first sum in the last line of (5.2.11) actually vanishes:

$$0 = \langle f, \phi_{0,m}\rangle_{L_2(\mathbb{R})} = \frac{c_{2m}^{(1)}(f) + c_{2m+1}^{(1)}(f)}{\sqrt{2}}, \qquad m \in \mathbb{Z},$$

where we have used the fact that $\{\psi_{0,k},\ \phi_{0,k}\ :\ k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal set. Hence $f \in W_0$ has the representation

$$f = \sum_{k\in\mathbb{Z}} \frac{c_{2k}^{(1)}(f) - c_{2k+1}^{(1)}(f)}{\sqrt{2}}\, \psi_{0,k},$$

which shows that $\{\psi_{0,k}\ :\ k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$-orthonormal basis for W_0. This proves (i). □

Exercise 63 *Proof the identities (5.2.7) and (5.2.8).*

Exercise 64 *In the proof of Lemma 5.11 it was claimed that it is enough to verify the property (i) for W_0 and the property (ii) for V_1, and that this then would imply the stated property (i) for any W_j and the property (ii) for V_{j+1}, due to the definition of the spaces V_j via scaling. Verify that this is true.*

Exercise 65 *Let $\phi(x) = \chi_{[0,1)}(x)$ be the Haar scaling function, and let $\psi(x) = \phi(2x) - \phi(2x-1)$ be the Haar wavelet. Let V_j, $j \in \mathbb{Z}$, denote the scale spaces and let W_j, $j \in \mathbb{Z}$, denote the detail spaces of the Haar scaling function and the Haar wavelet. Let*

$$f(x) = \begin{cases} 3 & \textit{if} \quad x \in \left[-\frac{1}{2}, 0\right), \\ 1 & \textit{if} \quad x \in \left[0, \frac{1}{2}\right), \\ -2 & \textit{if} \quad x \in \left[\frac{1}{2}, 1\right), \\ -1 & \textit{if} \quad x \in [1,2), \\ 2 & \textit{if} \quad x \in [3,4), \\ 0 & \textit{if} \quad x \in \mathbb{R} \setminus \left(\left[-\frac{1}{2}, 2\right) \cup [3,4)\right). \end{cases}$$

(a) Sketch the function f.

(b) Show that $f \in V_1$, and find the representation of f with respect to the $L_2(\mathbb{R})$-orthonormal basis $\{\phi_{1,k}(x) = \sqrt{2}\,\phi(2x-k)\ :\ k \in \mathbb{Z}\}$ of V_1.

(c) Derive the representation of the function f with respect to the $L_2(\mathbb{R})$-orthonormal basis $\{\phi_{0,k}(x) = \phi(x-k),\ \psi_{0,k}(x) = \psi(x-k)\ :\ k \in \mathbb{Z}\}$ of V_1.

Note that we can iterate the decomposition $V_{j+1} = V_j \oplus W_j$ of V_{j+1} into a direct sum. More precisely,

$$V_{j+1} = W_j \oplus V_j = W_j \oplus W_{j-1} \oplus V_{j-1} = \ldots = \overline{\bigoplus_{\ell \le j} W_\ell}^{\|\cdot\|_{L_2(\mathbb{R})}}.$$

Noting that $V_j \subset V_{j+1}$ and using Theorem 5.8 (iii), letting j tend to infinity shows that

$$L_2(\mathbb{R}) = \overline{\bigoplus_{j\in\mathbb{Z}} W_j}^{\|\cdot\|_{L_2(\mathbb{R})}}.$$

Analogously we can obtain all the other relations summarised in the lemma below.

Lemma 5.12 (orthogonal sums of Haar scale spaces and Haar detail spaces)
Let $\phi(x) = \chi_{[0,1)}(x)$ be the Haar scaling function, and let $\psi(x) = \phi(2x) - \phi(2x-1)$ be the Haar wavelet. Let V_j and W_j, $j \in \mathbb{Z}$, be the corresponding scale and detail spaces, as introduced in Definitions 5.6 and 5.10. Then the following orthogonal sum relations hold true:

(i) $V_{J+1} = V_{J_0} \oplus W_{J_0} \oplus W_{J_0+1} \oplus \cdots \oplus W_J = V_{J_0} \oplus \left(\bigoplus_{j=J_0}^{J} W_j \right)$ *for all* $-\infty < J_0 \leq J < \infty$.

(ii) $V_{J+1} = \overline{\bigoplus_{\ell=-\infty}^{J} W_\ell}^{\|\cdot\|_{L_2(\mathbb{R})}}$ *for all* $J \in \mathbb{Z}$.

(iii) $L_2(\mathbb{R}) = \overline{\bigoplus_{j\in\mathbb{Z}} W_j}^{\|\cdot\|_{L_2(\mathbb{R})}}$.

From Lemma 5.12 (iii) we can finally conclude that $\{\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k) \,:\, j,k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for $L_2(\mathbb{R})$.

Corollary 5.13 (Haar wavelet is an orthogonal wavelet)
Let $\phi(x) = \chi_{[0,1)}(x)$ be the Haar scaling function, and let $\psi(x) = \phi(2x) - \phi(2x-1)$ be the Haar wavelet. The set

$$M := \left\{\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k) \,:\, j,k \in \mathbb{Z}\right\}$$

*forms an $L_2(\mathbb{R})$-**orthonormal basis of** $L_2(\mathbb{R})$. Hence the Haar wavelet ψ is an **orthogonal wavelet**.*

Proof of Corollary 5.13: In Example 5.5 we have verified that M is an $L_2(\mathbb{R})$-orthonormal set. From Lemma 5.11 (i) we know that

$$\left\{\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k) \,:\, k \in \mathbb{Z}\right\}$$

is an $L_2(\mathbb{R})$-orthonormal basis for W_j. From the fact that V_{j+1} is the orthogonal sum $V_{j+1} = V_j \oplus W_j$ and from Lemma 5.11 (ii) we can conclude that M is an $L_2(\mathbb{R})$-orthonormal set. This, together with Lemma 5.12 (iii), implies that

$$\overline{M}^{\|\cdot\|_{L_2(\mathbb{R})}} = \overline{\left\{\psi_{j,k}(x) = 2^{j/2}\,\psi(2jx - k) \,:\, j,k \in \mathbb{Z}\right\}}^{\|\cdot\|_{L_2(\mathbb{R})}} = L_2(\mathbb{R}).$$

Hence M is an $L_2(\mathbb{R})$-orthonormal basis for $L_2(\mathbb{R})$. □

Lemma 5.12 (i) describes the orthogonal decomposition of a function into a basic approximation in V_{J_0} plus a sum of approximations in the spaces W_j, $j = J_0, J_0+1, \ldots, J$. More precisely, for $f \in V_{J+1}$, we have

$$f = \underbrace{\sum_{k\in\mathbb{Z}} c_k^{(J_0)}(f)\,\phi_{J_0,k}}_{=:\,P_{J_0}(f)} + \sum_{j=J_0}^{J} \underbrace{\sum_{k\in\mathbb{Z}} d_k^{(j)}(f)\,\psi_{j,k}}_{=:\,Q_j(f)}, \tag{5.2.12}$$

where the coefficients are given by

$$c_k^{(j)}(f) = \langle f, \phi_{j,k} \rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\, \overline{\phi_{j,k}(x)}\, \mathrm{d}x, \tag{5.2.13}$$

$$d_k^{(j)}(f) = \langle f, \psi_{j,k} \rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\, \overline{\psi_{j,k}(x)}\, \mathrm{d}x. \tag{5.2.14}$$

The formula (5.2.12) is at the heart of the **wavelet transform** that is discussed in Section 5.4.

The intuition for understanding formula (5.2.12) is as follows. The approximation $P_{J_0}(f)$ of f in V_{J_0} is a **coarse approximation** that captures the global trends of the signal f. We note that $P_{J_0}(f)$ is the $L_2(\mathbb{R})$-orthogonal projection of f onto the scale space V_{J_0} (and also the best approximation of f in V_{J_0}). As $P_{J_0}(f)$ is a 'coarse' approximation of f corresponding to long wavelength (low frequency) contributions of f, we can consider the orthogonal projection P_{J_0} as a **low-pass filter**.

Each approximation $Q_j(f)$, $j = J_0, \ldots, J$, describes **details of the signal** f, which are added to the original approximation. From the definition of the scaled versions of the Haar wavelet, it is intuitively clear that, the larger j, the finer the details that can be approximated by $Q_j(f)$. We note that $Q_j(f)$ is the $L_2(\mathbb{R})$-orthogonal projection of f onto the detail space W_j (and also the best approximation of f in W_j). In the language of signal processing we may interpret the operators Q_j, $j = J_0, J_0+1, \ldots, J$, as **band-pass filters**, which means that each $Q_j f$ approximates details in the signal f corresponding to a certain band of wavelengths (frequencies). The higher the index j the shorter the wavelengths (the higher the frequencies) in the signal that are approximated by $Q_j f$.

5.3 Multiresolution Analysis

Finally, we give the general definition of a multiresolution analysis (see Theorem 5.8 for the special case of the multiresolution analysis generated by the Haar scaling function).

Definition 5.14 (multiresolution analysis)
A family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces V_j of $L_2(\mathbb{R})$ is called a ***multiresolution analysis (MRA)*** *if the following properties are satisfied:*

(i) $V_j \subset V_{j+1}$ *for all* $j \in \mathbb{Z}$.

(ii) For any $j \in \mathbb{Z}$, *we have* $f \in V_j$ *if and only if* $f(2\cdot) \in V_{j+1}$.

(iii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j}^{L_2(\mathbb{R})} = L_2(\mathbb{R})$.

(iv) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.

(v) There exists a function $\phi \in L_2(\mathbb{R})$ *such that* $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ *is an* ***orthonormal basis*** *for* V_0 *equipped with the* $L_2(\mathbb{R})$ *inner product.*

The function ϕ in property (v) is called a ***scaling function*** *of the multiresolution analysis.*

Comparison with Theorem 5.8 shows that the scale spaces $\{V_j\}_{j\in\mathbb{Z}}$ of the the Haar scaling function form a multiresolution analysis in the sense of Definition 5.14 and that the Haar scaling function is the scaling function in property (v) of Definition 5.14.

We also note that Definition 5.14 does not specify that the scaling function is unique; indeed one multiresolution analysis, can have several possible scaling functions.

Let us extract a number of useful consequences from this definition.

Corollary 5.15 (derived properties of a multiresolution analysis)
Let a family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces V_j of $L_2(\mathbb{R})$ be a multiresolution analysis. Then the following holds true:

(i) For any $j \in \mathbb{Z}$, a function f belongs to V_j if and only if $f(2^{-j}\cdot)$ belongs to V_0.

(ii) For any $j \in \mathbb{Z}$, the set $\{\phi_{j,k} : k \in \mathbb{Z}\}$ of functions $\phi_{j,k}(x) := 2^{j/2}\,\phi(2^j x - k)$ forms an $L_2(\mathbb{R})$-orthonormal basis for V_j.

(iii) The scaling function itself is not uniquely determined.

Notation: From Corollary 5.15 (ii) it is clear that, given a scaling function ϕ for a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$, the spaces V_j of the multiresolution analysis can be described via

$$V_j = \overline{\text{span}\left\{\phi_{j,k}(x) := 2^{j/2}\,\phi(2^j x - k) \,:\, k \in \mathbb{Z}\right\}}^{\|\cdot\|_{L_2(\mathbb{R})}}.$$

Therefore we will say that **the scaling function ϕ generates the multiresolution analysis** $\{V_j\}_{j\in\mathbb{Z}}$. We will also refer to the spaces V_j as **scale spaces**.

Exercise 66 *Prove Corollary 5.15 (i) and (ii).*

As in the case of the Haar wavelet, we can decompose the closed space V_{j+1} into $V_{j+1} = V_j \oplus W_j$, where W_j is the orthogonal complement of V_j in V_{j+1}.

Definition 5.16 (detail spaces)
*Let a family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces V_j of $L_2(\mathbb{R})$ be a multiresolution analysis. The **detail space** W_j is defined as the $L_2(\mathbb{R})$-**orthogonal complement of** V_j **in** V_{j+1}, that is, we have the orthogonal sum*

$$V_{j+1} = V_j \oplus W_j.$$

Definition 5.16 immediately implies the the following lemma.

Lemma 5.17 (orthogonal sum decomposition of V_J and $L_2(\mathbb{R})$)
Let a family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces V_j of $L_2(\mathbb{R})$ be a multiresolution analysis, and let $\{W_j\}_{j\in\mathbb{Z}}$ denote the corresponding family of detail spaces. Then we have the following orthogonal sum relations:

(i) $V_{J+1} = V_{J_0} \oplus W_{J_0} \oplus W_{J_0+1} \oplus \ldots \oplus W_J = V_{J_0} \oplus \left(\bigoplus_{j=J_0}^{J} W_j \right)$ *for all* $-\infty < J_0 \le J < \infty$.

(ii) $V_{J+1} = \overline{\bigoplus_{j=-\infty}^{J} W_j}^{\|\cdot\|_{L_2(\mathbb{R})}}$ *for all* $J \in \mathbb{Z}$.

(iii) $L_2(\mathbb{R}) = \overline{\bigoplus_{j\in\mathbb{Z}} W_j}^{\|\cdot\|_{L_2(\mathbb{R})}}$.

Wavelet Decomposition/Reconstruction and Multiresolution Analysis: A multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ already gives us a wavelet analysis; even though we have not yet introduced a wavelet ψ whose scaled and shifted versions should provide $L_2(\mathbb{R})$-orthonormal bases for the detail spaces W_j. To get a wavelet decomposition of $f \in L_2(\mathbb{R})$, we introduce the $L_2(\mathbb{R})$**-orthogonal projection** $P_j : L_2(\mathbb{R}) \to V_j$ **onto the subspace** V_j. Then the $L_2(\mathbb{R})$**-orthogonal projection** $Q_j : L_2(\mathbb{R}) \to W_j$ **onto** W_j is given by $Q_j := P_{j+1} - P_j$. (This follows from the orthogonal sum $V_{j+1} = V_j \oplus W_j$.) Hence, we have a decomposition of the $L_2(\mathbb{R})$-orthogonal projection operator P_{j+1} as $P_{j+1} = P_j + Q_j$, and more generally (using this repeatedly)

$$P_{J+1} = P_{J_0} + \sum_{j=J_0}^{J} Q_j. \tag{5.3.1}$$

Thus the $L_2(\mathbb{R})$-orthogonal projection $P_{J+1}(f)$ of $f \in L_2(\mathbb{R})$ onto V_{J+1} can be decomposed/reconstructed as follows

$$P_{J+1}(f) = P_{J_0}(f) + \sum_{j=J_0}^{J} Q_j(f). \tag{5.3.2}$$

As in the case of the Haar scaling function and wavelet, we interpret $P_{J_0}f$ as a **basic/coarse (low-frequency) approximation** of the signal f and interpret the approximations $Q_j f$ as **(band-pass filtered) details** that contain more and more information on the finer details of the signal f as j increases. Both (5.3.1) and (5.3.2) can be seen as a formal way of writing down the wavelet decomposition.

The preceding definitions and statements raise **several questions:** How do we find a multiresolution analysis, or, maybe more practically minded, how do we find a scaling function ϕ that generates a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ via

$$V_j := \overline{\operatorname{span}\left\{\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \,:\, k \in \mathbb{Z}\right\}}^{\|\cdot\|_{L_2(\mathbb{R})}}, \qquad j \in \mathbb{Z},$$

and where $\{\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k) \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis of V_j? Also, once we have a scaling function, how do we find a wavelet ψ such that

$$\left\{\psi_{j,k}(x) = 2^{j/2}\,\psi(2^j x - k)\,:\,k \in \mathbb{Z}\right\}$$

forms an $L_2(\mathbb{R})$-orthonormal basis of the detail space W_j? Answering these questions is a highly non-trivial task that goes beyond the scope of this course and involves the (continuous) Fourier transform. The full analysis of the answers to these questions would need to be discussed in a follow-up course that solely focusses on wavelets.

To round up our discussion, we will collect a few properties of a scaling function of a multiresolution analysis. Then we will generalise and give conditions on a function $\phi \in L_2(\mathbb{R})$ such that the spaces

$$V_j := \overline{\operatorname{span}\left\{\phi_{j,k}(x) = 2^{j/2}\,\phi(2^j x - k)\,:\,k \in \mathbb{Z}\right\}}^{\|\cdot\|_{L_2(\mathbb{R})}}, \qquad j \in \mathbb{Z},$$

and the function ϕ satisfy conditions (i), (ii) and (v) in Definition 5.14. To derive conditions on ϕ that guarantee that also conditions (iii) and (iv) in Definition 5.14 are satisfied requires mathematical analysis that goes beyond the scope of this course.

Theorem 5.18 (refinement equation for the scaling function)
*A scaling function ϕ of a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ satisfies a **refinement equation** (or **two-scale relation**)*

$$\phi(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k\,\phi(2x - k) \tag{5.3.3}$$

with a sequence of coefficients $(h_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ (which depend on ϕ).

The refinement equation says that the scaling function $\phi \in V_0$ can be expressed as a $L_2(\mathbb{R})$-convergent expansion in terms of the basis functions $\phi_{1,k} = \sqrt{2}\,\phi(2x-k)$, $k \in \mathbb{Z}$, of V_1.

Let us first consider our example of the Haar scaling function again:

Example 5.19 (refinement equation of the Haar scaling function)
If $\phi(x) = \chi_{[0,1)}(x)$ is the Haar scaling function then we have (see (5.2.6) in Lemma 5.9)

$$\phi(x) = \phi(2x) + \phi(2x-1) = \sqrt{2}\left(\frac{1}{\sqrt{2}}\,\phi(2x) + \frac{1}{\sqrt{2}}\,\phi(2x-1)\right),$$

and hence we have (5.3.3) with the sequence $(h_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ given by $h_0 = h_1 = 1/\sqrt{2}$ and $h_k = 0$ for all $k \in \mathbb{Z}\setminus\{0,1\}$.

Proof of Theorem 5.18: Since $\phi \in V_0 \subset V_1$ and since

$$\{\phi_{1,k}(x) := 2^{1/2}\phi(2^j x - k)\,:\,k \in \mathbb{Z}\}$$

is an $L_2(\mathbb{R})$-orthonormal basis of V_1, we have

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k\,\phi_{1,k}(x) = \sqrt{2}\sum_{k\in\mathbb{Z}} h_k\,\phi(2\,x-k).$$

with the coefficients $h_k := \langle \phi, \phi_{1,k} \rangle_{L_2(\mathbb{R})}$, $k \in \mathbb{Z}$. Furthermore from Parseval's identity for V_1,

$$\|(h_k)_{k\in\mathbb{Z}}\|_2 = \left(\sum_{k\in\mathbb{Z}} |h_k|^2 \right)^{1/2} = \|\phi\|_{L_2(\mathbb{R})} < \infty,$$

and we know that $(h_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$. □

Now we want to go away from the assumption that ϕ is the scaling function of a multiresolution analysis. Instead we want to establish assumptions on $\phi \in L_2(\mathbb{R})$ that give us a refinement equation and properties (i), (ii), and (v) in the definition of a multiresolution analysis for the spaces $\{V_j\}_{j\in\mathbb{Z}}$, defined by

$$V_j := \overline{\text{span}\left\{ \phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \,:\, k \in \mathbb{Z} \right\}}^{\|\cdot\|_{L_2(\mathbb{R})}}.$$

Theorem 5.20 (consequences of assumptions on $\phi \in L_2(\mathbb{R})$)
Let $\phi \in L_2(\mathbb{R})$ satisfy the condition that $\left\{\phi_{0,k}(x) = \phi(x-k) \,:\, k \in \mathbb{Z}\right\}$ is an $L_2(\mathbb{R})$-orthonormal set. Define the spaces V_j, $j \in \mathbb{Z}$, by

$$V_j := \overline{\text{span}\left\{ \phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \,:\, k \in \mathbb{Z} \right\}}^{\|\cdot\|_{L_2(\mathbb{R})}}.$$

Then the following holds true:

(i) For any $j \in \mathbb{Z}$, the set $\left\{\phi_{j,k}(x) := 2^{j/2}\phi(2^j x - k) \,:\, k \in \mathbb{Z}\right\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_j.

(ii) $f \in V_j$ if and only if $f(2\cdot) \in V_j$ for all $j \in \mathbb{Z}$.

(iii) The function ϕ is in V_1 if and only if ϕ satisfies a refinement equation

$$\phi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k \, \phi(2x - k) \tag{5.3.4}$$

with a sequence $(h_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$.

(iv) If $\phi \in V_1$, then $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$.

Before we prove the theorem, we state the following corollary.

Corollary 5.21 (conditions on $\phi \in L_2(\mathbb{R})$ for (i), (ii), (v) in Definition 5.14)
Let $\phi \in L_2(\mathbb{R})$ satisfy the assumption that $\{\phi_{0,k}(x) = \phi(x-k) \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal set, and define the spaces V_j, $j \in \mathbb{Z}$,

$$V_j := \overline{\text{span}\left\{ \phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k) \,:\, k \in \mathbb{Z} \right\}}^{\|\cdot\|_{L_2(\mathbb{R})}}.$$

Assume further that $\phi \in V_1$. Then the set $\{V_j\}_{j\in\mathbb{Z}}$ of subspaces of $L_2(\mathbb{R})$ and the function ϕ satisfy conditions (i), (ii), and (v) from Definition 5.14 of a multiresolution analysis.

Proof of Corollary 5.21: The corollary follows essentially from Theorem 5.20. Since the set

$\{\phi_{0,k}(x) = \phi(x-k) \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal set, it is clear, from the definition of V_0, that $\{\phi_{0,k}(x) = \phi(x-k) \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis of V_0. Hence (v) in Definition 5.14 is satisfied. Since $\phi \in V_1$, we find from Theorem 5.20 (iv) that (i) in Definition 5.14 is satisfied. That property (ii) in Definition 5.14 is satisfied follows immediately from Theorem 5.20 (ii). □

Proof of Theorem 5.20: We verify statements (i) to (iv).

(i) With the substitution $y = 2^j x,\ \mathrm{d}y = 2^j\,\mathrm{d}x$, we obtain

$$\langle \phi_{j,k}, \phi_{j,\ell}\rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} 2^j\, \phi(2^j x - k)\, \overline{\phi(2^j x - \ell)}\,\mathrm{d}x = \int_{\mathbb{R}} \phi(y-k)\,\overline{\phi(y-\ell)}\,\mathrm{d}y = \delta_{k,\ell}.$$

Hence $\{\phi_{j,k} \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal set. By the definition of V_j it is clear that this $L_2(\mathbb{R})$-orthonormal set is an $L_2(\mathbb{R})$-orthonormal basis for V_j.

(ii) $\Rightarrow$: Assume that $f \in V_j$. Since $\{\phi_{j,k} \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_j, there exists a sequence $(c_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that

$$f(x) = \sum_{k\in\mathbb{Z}} c_k\, \phi_{j,k}(x) = \sum_{k\in\mathbb{Z}} c_k\, 2^{j/2}\, \phi(2^j x - k).$$

Thus the function $g(x) := f(2x)$ is given by

$$g(x) = f(2x) = \sum_{k\in\mathbb{Z}} c_k\, \phi_{j,k}(2x) = \sum_{k\in\mathbb{Z}} \frac{c_k}{\sqrt{2}}\, 2^{(j+1)/2}\, \phi(2^{j+1}x - k) = \sum_{k\in\mathbb{Z}} \frac{c_k}{\sqrt{2}}\, \phi_{j+1,k}(x),$$

and the sequence $(c_k/\sqrt{2})_{k\in\mathbb{Z}}$ is also in $\ell_2(\mathbb{Z})$. As the set $\{\phi_{j+1,k} \,:\, k \in \mathbb{Z}\}$ forms an $L_2(\mathbb{R})$-orthonormal basis of V_{j+1}, clearly $g \in V_{j+1}$.

$\Leftarrow$: Assume that $g = f(2\cdot)$ is in V_{j+1}. Since $\{\phi_{j+1,k} \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_{j+1}, there exists a sequence $(c_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that

$$g(x) = \sum_{k\in\mathbb{Z}} c_k\, \phi_{j+1,k}(x) = \sum_{k\in\mathbb{Z}} c_k\, 2^{(j+1)/2}\, \phi(2^{j+1}x - k).$$

Substituting $x = y/2$, we find

$$f(y) = g(y/2) = \sum_{k\in\mathbb{Z}} \sqrt{2}\, c_k\, 2^{j/2}\, \phi(2^j y - k). = \sum_{k\in\mathbb{Z}} \sqrt{2}\, c_k\, \phi_{j,k}(y).$$

As $\{\phi_{j,k} \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_j and as $(\sqrt{2}\, c_k)_{k\in\mathbb{Z}}$ is in $\ell_2(\mathbb{Z})$ (because $(c_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$), the function $f(y) = g(y/2)$ is clearly in V_j.

(iii) $\Rightarrow$: Assume that $\phi \in V_1$. Since $\{\phi_{1,k} \,:\, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_1, there exists a sequence $(h_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k\, \phi_{1,k}(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k\, \phi(2x - k),$$

which is just the refinement equation (5.3.4).

$\Leftarrow$: Conversely, assume that ϕ satisfies a refinement equation

$$\phi(x) = \sqrt{2} \sum_{k\in\mathbb{Z}} h_k \, \phi(2x-k) = \sum_{k\in\mathbb{Z}} h_k \, \phi_{1,k}(x), \tag{5.3.5}$$

with some sequence $(h_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$. Since $\{\phi_{1,k} \, : \, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_1, (5.3.5) just means that $\phi \in V_1$.

(iv) Let $f \in V_j$. Since $\{\phi_{j,k} \, : \, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_j, there exists a sequence $(c_k)_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that in the $L_2(\mathbb{R})$-sense

$$f(x) = \sum_{k\in\mathbb{Z}} c_k \, \phi_{j,k}(x) = \sum_{k\in\mathbb{Z}} c_k \, 2^{j/2} \, \phi(2^j x - k). \tag{5.3.6}$$

First we show that the functions $\phi_{j,\ell}$, $\ell \in \mathbb{Z}$, are in V_{j+1}. Indeed, since $\phi \in V_1$, the refinement equation (5.3.4) holds true. Letting in the refinement equation $x = 2^j y - \ell$ and multiplying the equation with $2^{j/2}$ yields

$$2^{j/2} \, \phi(2^j y - \ell) = \sum_{k\in\mathbb{Z}} h_k \, 2^{(j+1)/2} \, \phi\big(2(2^j y - \ell) - k\big) = \sum_{k\in\mathbb{Z}} h_k \, 2^{(j+1)/2} \, \phi\big(2^{j+1} y - (k+2\ell)\big)$$

or equivalently

$$\phi_{j,\ell} = \sum_{k\in\mathbb{Z}} h_k \, \phi_{j+1,k+2\ell}. \tag{5.3.7}$$

Since $(h_k)_{k\in\mathbb{Z}}$ is in $\ell_2(\mathbb{Z})$ and since $\{\phi_{j+1,k} \, : \, k \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal basis for V_{j+1}, (5.3.7) implies that $\phi_{j,\ell} \in V_{j+1}$ for all $\ell \in \mathbb{Z}$. As the set $\{\phi_{j,\ell} \, : \, \ell \in \mathbb{Z}\}$ is an $L_2(\mathbb{R})$-orthonormal set in V_{j+1}, we see from (5.3.6) that the function $f \in V_j$ is also in V_{j+1}. Since $f \in V_j$ was arbitrary, we have $V_j \subset V_{j+1}$.

This completes the proof. $\square$

5.4 The Wavelet Transform

Now we analyse the **decomposition and reconstruction process** described in Lemma 5.12 for the multiresolution analysis generated by the Haar scaling function and in Lemma 5.17 for the multiresolution analysis generated by an arbitrary scaling function ϕ.

Assumption: For the case of a general multiresolution analysis, as defined in Definition 5.14, we assume in this section that we have also constructed a wavelet ψ such that, for any $j \in \mathbb{Z}$, the set

$$\big\{\psi_{j,k}(x) := 2^{j/2} \, \psi(2^j x - k) \, : \, k \in \mathbb{Z}\big\}$$

forms an $L_2(\mathbb{R})$-orthonormal basis of the detail space W_j, and such that

$$\big\{\psi_{j,k}(x) := 2^{j/2} \, \psi(2^j x - k) \, : \, j,k \in \mathbb{Z}\big\}$$

forms an $L_2(\mathbb{R})$-orthonormal basis for $L_2(\mathbb{R})$.

You should think in this section of the Haar scaling function and the Haar wavelet, since these are easily visualised and make the procedure more intuitive.

Suppose, we are given a function $f \in L_2(\mathbb{R})$. Then, we can fix a level $J \in \mathbb{Z}$ and consider the $L_2(\mathbb{R})$**-orthogonal projection** $P_J(f)$ **of** f **onto** V_J, given by

$$P_J(f) = \sum_{k\in\mathbb{Z}} c_k^{(J)}(f)\,\phi_{J,k},$$

where the coefficients $c_k^{(J)}(f)$, $k \in \mathbb{Z}$, are given by

$$c_k^{(J)}(f) = \langle f, \phi_{J,k}\rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\,\overline{\phi_{J,k}(x)}\,\mathrm{d}x = 2^{J/2}\int_{\mathbb{R}} f(x)\,\overline{\phi(2^J x - k)}\,\mathrm{d}x.$$

But instead of storing all the relevant Fourier coefficients $c_k^{(J)}$, $k \in \mathbb{Z}$, of the approximation in V_J, we can employ the **decomposition**

$$\begin{aligned} V_J &= V_{J-1} \oplus W_{J-1}, \\ (\text{approximation in } V_J) &= (\text{coarse approximation in } V_{J-1}) \;\oplus\; (\text{details in } W_{J-1}), \end{aligned}$$

to store the coefficients of the **approximation at the coarser level** V_{J-1} and the coefficients of the **details in** W_{J-1}.

$$P_J(f) = \underbrace{\sum_{k\in\mathbb{Z}} c_k^{(J-1)}(f)\,\phi_{J-1,k}}_{=:\,P_{J-1}(f)} + \underbrace{\sum_{k\in\mathbb{Z}} d_k^{(J-1)}(f)\,\psi_{J-1,k}}_{=:\,Q_{J-1}(f)}, \tag{5.4.1}$$

where the coefficients are given by

$$\begin{aligned} c_k^{(J-1)}(f) &= \langle f, \phi_{J-1,k}\rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\,\overline{\phi_{J-1,k}(x)}\,\mathrm{d}x, \\ d_k^{(J-1)}(f) &= \langle f, \psi_{J-1,k}\rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\,\overline{\psi_{J-1,k}(x)}\,\mathrm{d}x. \end{aligned}$$

We note that $P_{J-1} : L_2(\mathbb{R}) \to V_{J-1}$ is the $L_2(\mathbb{R})$-orthogonal projection onto V_{J-1}, and that $Q_{J-1} : L_2(\mathbb{R}) \to W_{J-1}$ is the $L_2(\mathbb{R})$**-orthogonal projection onto** W_{J-1}. As the direct sum $V_J = V_{J-1} \oplus W_{J-1}$ is an orthogonal sum, we have $P_J = P_{J-1} + Q_{J-1}$ and Q_{J-1} can also be described by $Q_{J-1} := P_J - P_{J-1}$. Repeating this process again and again for the approximations $P_j f$ in the scale spaces V_j, $j = J-1, J-2, \ldots, J_0+1$, we find

$$P_J(f) = \underbrace{\sum_{k\in\mathbb{Z}} c_k^{(J_0)}(f)\,\phi_{J_0,k}}_{=:\,P_{J_0}(f)} + \sum_{j=J_0}^{J-1} \underbrace{\sum_{k\in\mathbb{Z}} d_k^{(j)}(f)\,\psi_{j,k}}_{=:\,Q_j(f)}, \tag{5.4.2}$$

where $Q_j : L_2(\mathbb{R}) \to W_j$ is the $L_2(\mathbb{R})$-orthogonal projection onto the detail space W_j and where the coefficients are given by

$$c_k^{(j)}(f) = \langle f, \phi_{j,k}\rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\,\overline{\phi_{j,k}(x)}\,\mathrm{d}x, \tag{5.4.3}$$

$$d_k^{(j)}(f) = \langle f, \psi_{j,k} \rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\, \overline{\psi_{j,k}(x)}\, \mathrm{d}x. \tag{5.4.4}$$

This **decomposition process** is indicated in Figure 5.1, where $c^{(j)} = \left(c_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$ denotes the coefficient sequence of $P_j(f)$, defined by (5.4.3) and $d^{(j)} = \left(d_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$ denotes the coefficient sequence of $Q_j(f)$, defined by (5.4.4).

$$\begin{array}{ccccccccccc}
 & & d^{(J-1)} & & d^{(J-2)} & & & & d^{(J_0+1)} & & d^{(J_0)} \\
 & \nearrow & & \nearrow & & \nearrow & & \nearrow & & \nearrow & \\
c^{(J)} & \longrightarrow & c^{(J-1)} & \longrightarrow & c^{(J-2)} & \longrightarrow & \cdots\; c^{(J_0+2)} & \longrightarrow & c^{(J_0+1)} & \longrightarrow & c^{(J_0)}
\end{array}$$

Figure 5.1: Schematic representation of the wavelet decomposition.

The decomposition scheme in Figure 5.1 is in general referred to as the **wavelet decomposition**. Naturally, this process can be reversed resulting in the **wavelet reconstruction**. Here, we start on a very coarse level and add details to the approximation to derive a more detailed version. This is also described by formula (5.4.2) and is schematically shown in Figure 5.2.

$$\begin{array}{ccccccccccc}
d^{(J_0)} & & d^{(J_0+1)} & & d^{(J_0+2)} & & & & d^{(J-1)} & & \\
 & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\
c^{(J_0)} & \longrightarrow & c^{(J_0+1)} & \longrightarrow & c^{(J_0+2)} & \longrightarrow & c^{(J_0+3)}\; \cdots & \longrightarrow & c^{(J-1)} & \longrightarrow & c^{(J)}
\end{array}$$

Figure 5.2: Schematic representation of the wavelet reconstruction.

Wavelet decomposition and wavelet reconstruction together are referred to as the **wavelet transform**.

For mathematical and computational purposes, it remains to determine the relation between the coefficient sequence $\left(c_k^{(j+1)}(f)\right)_{k\in\mathbb{Z}}$ of $P_{j+1}(f) \in V_{j+1}$ and the coefficient sequence $\left(c_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$ of $P_j(f) \in V_j$ and $\left(d_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$ of $Q_j(f) \in W_j$. The formulas for computing $\left(c_k^{(j+1)}(f)\right)_{k\in\mathbb{Z}}$ from $\left(c_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$ and $\left(d_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$, and vice versa, naturally depend on the given scaling function and wavelet.

We will derive the formulas for computing $\left(c_k^{(j+1)}(f)\right)_{k\in\mathbb{Z}}$ from $\left(c_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$ and $\left(d_k^{(j)}(f)\right)_{k\in\mathbb{Z}}$, and vice versa, for the **Haar scaling function** and **Haar wavelet**.

To do this, recall that from $V_{j+1} = V_j \oplus W_j$, the function $P_{j+1}(f) \in V_{j+1}$ has the following two representations

$$P_{j+1}(f) \;=\; \sum_{k\in\mathbb{Z}} c_k^{(j+1)}(f)\, \phi_{j+1,k}$$

$$P_{j+1}(f) \;=\; \sum_{k\in\mathbb{Z}} c_k^{(j)}(f)\,\phi_{j,k} + \sum_{k\in\mathbb{Z}} d_k^{(j)}(f)\,\psi_{j,k}.$$

Then, for the wavelet decomposition, we have to compute $\big(c_k^{(j)}(f)\big)_{k\in\mathbb{Z}}$ and $\big(d_k^{(j)}(f)\big)_{k\in\mathbb{Z}}$ from $\big(c_k^{(j+1)}(f)\big)_{k\in\mathbb{Z}}$ and for the wavelet reconstruction, we have to compute $\big(c_k^{(j+1)}(f)\big)_{k\in\mathbb{Z}}$ from $\big(c_k^{(j)}(f)\big)_{k\in\mathbb{Z}}$ and $d_k^{(j)}(f)\big)_{k\in\mathbb{Z}}$.

Replacing in (5.2.7) and (5.2.8) x by $2^j x - k$ yields

$$\begin{aligned}\phi(2^j x - k) + \psi(2^j x - k) &= 2\,\phi(2^{j+1}x - 2k),\\ \phi(2^j x - k) - \psi(2^j x - k) &= 2\,\phi(2^{j+1}x - 2k - 1),\end{aligned}$$

or equivalently

$$\phi_{j+1,2k} = \frac{1}{\sqrt{2}}\big(\phi_{j,k} + \psi_{j,k}\big), \tag{5.4.5}$$

$$\phi_{j+1,2k+1} = \frac{1}{\sqrt{2}}\big(\phi_{j,k} - \psi_{j,k}\big). \tag{5.4.6}$$

Solving in (5.4.5) and (5.4.6) for $\phi_{j,k}$ and $\psi_{j,k}$, respectively, yields

$$\phi_{j,k} = \frac{1}{\sqrt{2}}\big(\phi_{j+1,2k} + \phi_{j+1,2k+1}\big), \tag{5.4.7}$$

$$\psi_{j,k} = \frac{1}{\sqrt{2}}\big(\phi_{j+1,2k} - \phi_{j+1,2k+1}\big). \tag{5.4.8}$$

From formulas (5.4.7) and (5.4.8), we obtain immediately that the formulas for the wavelet decomposition of the coefficients are given by

$$c_k^{(j)}(f) = \frac{1}{\sqrt{2}}\left(c_{2k}^{(j+1)}(f) + c_{2k+1}^{(j+1)}(f)\right), \tag{5.4.9}$$

$$d_k^{(j)}(f) = \frac{1}{\sqrt{2}}\left(c_{2k}^{(j+1)}(f) - c_{2k+1}^{(j+1)}(f)\right). \tag{5.4.10}$$

Indeed, taking in (5.4.7) and (5.4.8) the $L_2(\mathbb{R})$ inner product with f yields

$$\begin{aligned}c_k^{(j)}(f) &= \langle f, \phi_{j,k}\rangle_{L_2(\mathbb{R})}\\ &= \frac{1}{\sqrt{2}}\Big(\langle f, \phi_{j+1,2k}\rangle_{L_2(\mathbb{R})} + \langle f, \phi_{j+1,2k+1}\rangle_{L_2(\mathbb{R})}\Big)\\ &= \frac{1}{\sqrt{2}}\left(c_{2k}^{(j+1)}(f) + c_{2k+1}^{(j+1)}(f)\right),\\ d_k^{(j)}(f) &= \langle f, \psi_{j,k}\rangle_{L_2(\mathbb{R})}\\ &= \frac{1}{\sqrt{2}}\Big(\langle f, \phi_{j+1,2k}\rangle_{L_2(\mathbb{R})} - \langle f, \phi_{j+1,2k+1}\rangle_{L_2(\mathbb{R})}\Big)\\ &= \frac{1}{\sqrt{2}}\left(c_{2k}^{(j+1)}(f) - c_{2k+1}^{(j+1)}(f)\right),\end{aligned}$$

which verifies (5.4.9) and (5.4.10).

Likewise taking in (5.4.5) and (5.4.6) the $L_2(\mathbb{R})$ inner product with f, shows that the wavelet reconstruction formulas for the coefficients are given by

$$c_{2k}^{(j+1)}(f) = \frac{1}{\sqrt{2}}\left(c_k^{(j)}(f) + d_k^{(j)}(f)\right), \tag{5.4.11}$$

$$c_{2k+1}^{(j+1)}(f) = \frac{1}{\sqrt{2}}\left(c_k^{(j)}(f) - d_k^{(j)}(f)\right). \tag{5.4.12}$$

The formulas (5.4.11) and (5.4.12) could also have been derived by solving (5.4.9) and (5.4.10) for $c_{2k}^{(j+1)}(f)$ and $c_{2k+1}^{(j+1)}(f)$.

An **example of a wavelet decomposition** and **wavelet reconstruction** with the Haar wavelet is shown in Figure 5.3. The first row shows on the right the original function and on the left its approximation $P_5(f)$ in V_5. The next row shows the decomposition of $P_5(f)$ in its coarser part $P_4(f) \in V_4$ on the left and its details $Q_4(f) \in W_4$ on the right. The third row shows the decomposition of $P_4(f)$ into its coarser part $P_3(f) \in V_3$ on the right and its details $Q_3(f) \in W_3$ on the left, and so on.

Concluding comments: Finally, we want to give some thoughts to why the concept of wavelets is superior to the concept of the Fourier series. To explain this, we will think of the function $f \in L_2(\mathbb{R})$ as a signal of the variable x which could be the time. If we assume that our signal f is periodic with period 2π, then we can use the truncated Fourier series to approximate the function f on the interval $[-\pi, \pi]$, giving an approximation of the form

$$S_n(f)(x) := \frac{1}{\sqrt{2\pi}} \sum_{k=-n}^{n} \widehat{f}_k \, e^{ikx},$$

with the Fourier coefficients $\widehat{f}_k$ given by

$$\widehat{f}_k = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) \, e^{-ikx} \, \mathrm{d}x.$$

Each of the functions e^{ikx} has support $\operatorname{supp}(e^{ikx}) = \mathbb{R}$, and therefore, for computing $\widehat{f}_k$, information on f is needed on the whole interval $[-\pi, \pi]$. Due to their lack of localisation the functions e^{ikx} are ill suited for approximating local features of the signal (features that occur only on a certain subinterval of $[-\pi, \pi]$). In contrast the scaled versions on the Haar wavelet and the Haar scaling function have compact local support and for the evaluation of the coefficients (5.2.13) and (5.2.14) only information on f on the compact local support of $\phi_{j,k}$ and $\psi_{j,k}$ is needed. – From the numerical example illustrated in Figure 5.3 it is clear that the Haar wavelet is not an 'ideal' wavelet due to its lack of smoothness (it is not even continuous). However, there are other more complicated wavelets with compact support that offer the same localisation advantages as the Haar wavelet but provide a better approximation quality.

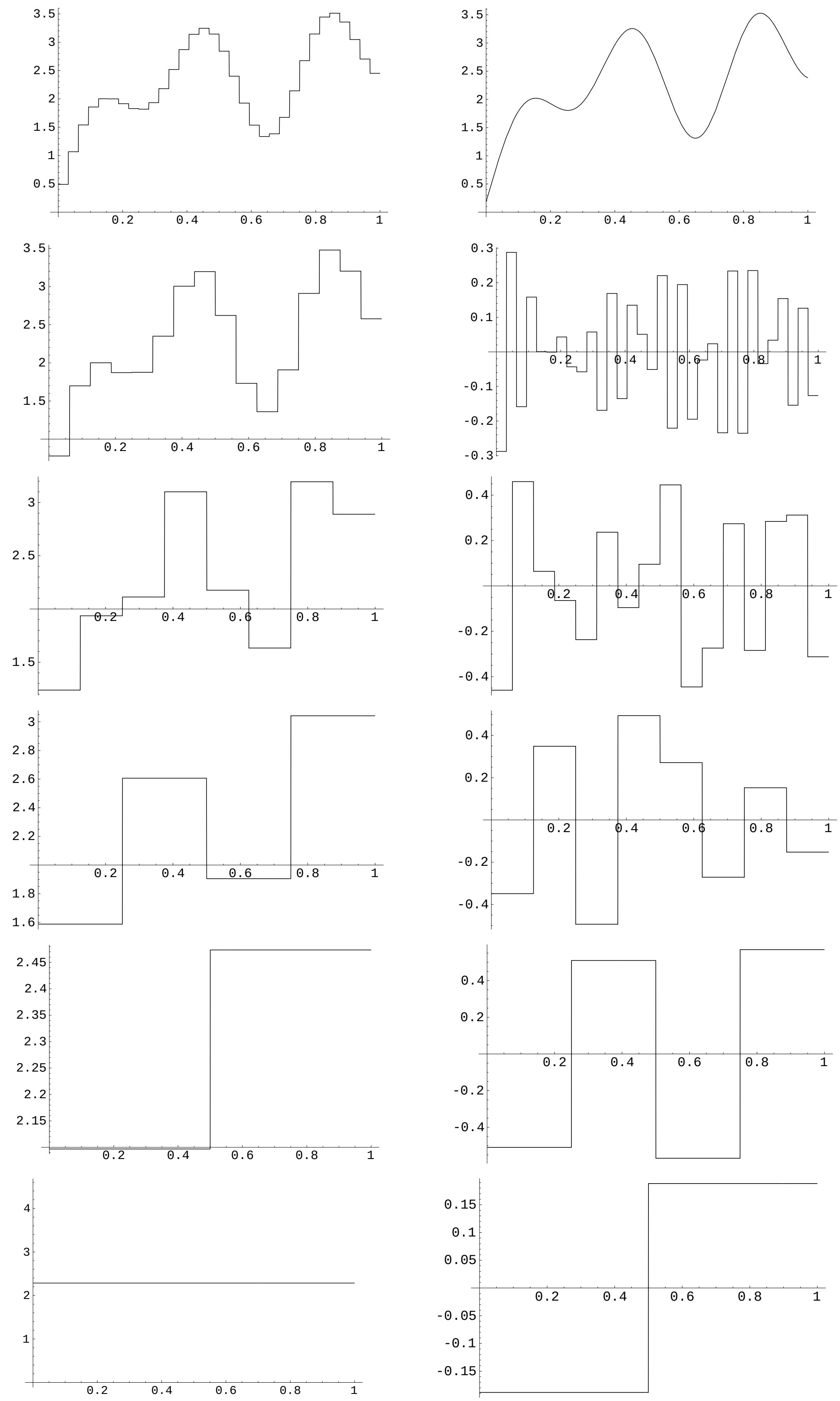

Figure 5.3: Wavelet decomposition of the function f plotted in the first row on the right: The left column includes the approximations $P_j(f)$ in the scale spaces V_j, where $j = 0, 1, 2, 3, 4, 5$ from bottom to top. The right column (apart from the top row) contains the approximations $Q_j(f)$ in the detail spaces W_j, where $j = 0, 1, 2, 3, 4$ from bottom to top. Remember that $P_{j+1}(f) = P_j(f) + Q_j(f)$.